Spring 2022, CS 586/IE 519: Combinatorial Optimization
Homework 1
Due: Thursday, Feb 3rd, 2022

Instructions and Policy: Each person should write up their own solutions independently. You need to indicate the names of the people you discussed a problem with. Solutions to most of these problems can be found from one source or the other. Try to solve on your own first, and cite your sources if you do use them.

Please write clearly and concisely. Refer to known facts. You should try to convince us that you know the solution, as quickly as possible.

Submit solutions to Problems 2, 4, 6. And try so solve others but do not submit.

Problem 1 Let $G = (V, E)$ be a directed graph and let $\ell : E \to \mathbb{R}$ be edge lengths (which can be negative). A directed cycle C in G is a negative length cycle if $\sum_{e \in C} \ell(e) < 0$. How can we convince ourselves that G does not have a negative length cycle? We say that a function $\pi : V \to \mathbb{R}$ is a valid potential if for each edge $(u, v) \in E$ we have $\pi(v) \leq \pi(u) + \ell(u, v)$.

- Prove that if G has a negative length cycle than there is no valid potential.
- Prove that if there is a valid potential then there is no negative length cycle in G.
- How can you use the Bellman-Ford algorithm to obtain a valid potential if G does not have a negative length cycle?
- Suppose you have computed a valid potential π in G. For each edge (u, v) define the reduced cost/length of (u, v) to be $\ell'(u, v) = \ell(u, v) + \pi(u) - \pi(v)$ which is non-negative. Suppose you want to find shortest path lengths from a node s to every other node. Show how you can compute shortest paths using the reduced costs via Dijkstra's algorithm and obtain the actual shortest paths in G. Thus, once we have potentials, finding single-source shortest paths can be reduced to the setting of non-negative edge lengths.

Problem 2 Let $G = (V, E)$ be an undirected graph. We use $\delta(A)$ to denote that set of edges with one end point in A and the other in $V - A$. For two disjoint sets of vertices $A, B \subset V$ we use $E(A, B)$ to denote the set of edges with one end point in A and another in B.

- Prove that for any $A, B \subseteq V$,
$$|\delta(A)| + |\delta(B)| = |\delta(A \cap B)| + |\delta(A \cup B)| + 2|E(A - B, B - A)|$$
- A real-valued set function $f : 2^V \to \mathbb{R}$ is called submodular iff $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq V$. Prove that for any graph $G = (V, E)$ the function $f(A) = |\delta(A)|$ is submodular. Note that f is also symmetric.

- Suppose $G = (V, E)$ be a directed graph and now consider the function $f(A) = |\delta^+(A)|$ where $|\delta^+(A)|$ is the number of edges that leave A (with the tail in A and head in $V - A$). Prove that f is submodular.

- **Not to submit:** A hypergraph $G = (V, E)$ consists of a finite vertex set V and a set of hyperedges E where each hyperedge $e \in E$ is a subset of V, that is $e \subseteq V$. Graphs are the special case when each hyperedge has two vertices. For a given hypergraph G and a set $A \subseteq V$ let $\delta(A)$ denote the set of hyperedges that contain (at least) one node in A and (at least) one in $V - A$. Prove that $f(A) = |\delta(A)|$ is submodular.

Problem 3 Let $G = (V, E)$ be an undirected graph. Let u, v, w be three distinct nodes. Describe an efficient algorithm to check if there is a simple path from u to v that contains w. What is the running time of your algorithm? *Hint:* Reduce to disjoint paths/flow. Note that the problem in directed graphs is NP-Complete!

Problem 4 Let $G = (V, E)$ be a directed graph with integer edge capacities, and let s, t be distinct nodes. The Ford-Fulkerson augmenting path algorithm finds a maximum flow f and proof of the maxflow-mincut theorem is based on showing the following. Suppose there is no s-t path in the residual graph G_f. Let A be the set of nodes reachable from s in the residual graph G_f. The proof shows that in this case $|\delta^+(A)| = |f|$ and hence we have optimality of the flow and the cut. It is easy to observe that a graph can have many s-t mincuts. However one can prove that there is a *unique* minimal s-t cut and a unique maximal mincut. To be precise we say that a set S is a minimal mincut if there is no S' such that $S' \subset S$ and S' is also a mincut. Minimality does not necessarily imply uniqueness since there can be two mincuts S, S' such that neither is a subset of the other.

- Prove that if A and B are s-t mincuts then $A \cap B$ and $A \cup B$ are also s-t minimum cuts. You can use submodularity or prove it in other ways.

- Argue that if f is a maximum flow then the set of reachable nodes from s in G_f is the unique minimal s-t cut.

- How would you find the unique maximal s-t cut given a maximum flow f?

Problem 5 Let $G = (V, E)$ be a d-regular bipartite graph.

- Prove that G has a perfect matching. You can use flows or Hall's theorem or any other method.

- Prove that G can be edge-colored with d colors. By an edge-coloring we mean a coloring of edges such that no two edges that intersect at a node have the same color.

- Prove that every bipartite graph with maximum degree d can be edge-colored with d colors.

- Give a simple example of a 2-regular non-bipartite graph that cannot be edge-colored with 2 colors.

Problem 6 Let $G = (V, E)$ be a directed graph with edge lengths/costs $c : E \to \mathbb{Z}$ (could be negative). Let s, t be distinct nodes in G. There are several ways to write an LP relaxation for the s-t shortest path. One way is to view an s-t shortest walks as a minimum cost set of edges in G that connect s to t. Letting x_e denote a variable for each edge $e \in E$ we express connectivity by asking for at least one edge to cross any cut $(S, V - S)$ such that $s \in S, t \notin S$.

$$\begin{aligned} \min \sum_{e \in E} c_e x_e & \\ \sum_{e \in \delta^+(S)} x_e \geq 1 & \quad s \in S, t \notin S \\ x_e \geq 0 & \quad e \in E \end{aligned}$$

Note that the LP has an exponential number of constraints.

- Argue that the LP is feasible iff there is an s-t path in G.
- Write the dual LP.
- Suppose all the edge lengths are non-negative. Argue that there is a feasible solution to the dual of value equal to the shortest path distance from s to t. *Hint:* Use shortest path distances from s to find a setting of dual values.