| CS 583: Approximation Algorithms | Lecture date: February 17, 2016 |
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The purpose of these notes is to explain the simple implications of the rank lemma about vertex solutions for linear programs. Recall that a *polyhedron* in $\mathbb{R}^n$ is defined as the intersection of finite collection of half spaces. Without loss of generality we can assume that it is defined by a system of inequalities of the form $Ax \leq b$ where $A$ is a $m \times n$ matrix and $b$ is a $m \times 1$ vector. A polyhedron $P$ is bounded if $P$ is contained in finite radius ball around the origin. A *polytope* in $\mathbb{R}^n$ is defined as the convex hull of a finite collection of points. A fundamental theorem about linear programming states that any bounded polyhedron is a polytope. If the polyhedron is not bounded then it can be expressed as the Minkowski sum of a polytope and a cone.

A bounded polyhedron $P$ in $\mathbb{R}^n$ defined by a system $Ax \leq b$ must necessarily have $m \geq n$. A point $p \in P$ is a basic feasible solution or a vertex solution of the system if it is the unique solution to a system $A'y = b'$ where $A'$ is a sub-matrix of $A$ with $n$ inequalities and the rank of $A'$ is equal to $n$. The inequalities in $A'$ are said to be *tight* for $y$. Note that there may be many other inequalities in $Ax \leq b$ that are tight at $y$ and in general there many be many different rank $n$ sub-matrices that give rise to the same basic feasible solution $y$.

**Lemma 1** *Suppose $y$ is a basic feasible solution of a system $Ax \leq b, \ell \leq x \leq u$ where $A$ is a $m \times n$ matrix and $\ell$ and $u$ are vectors defining lower and upper bounds on the variables $x \in \mathbb{R}^n$. Let $S = \{i : \ell_i < y_i < u_i\}$ be the set of indices of "fractional" variables in $y$. Then $|S| \leq rank(A) \leq m$. In particular the number of fractional variables in $y$ is at most the number of "non-trivial" constraints (those that are defined by $A$).*

The lemma is a simple consequence of the definition of basic feasible solution. It is interesting only when $rank(A)$ or $m$ is smaller than $n$, otherwise the claim is trivial. Before we prove it formally we observe some simple corollaries. Suppose we have a system $Ax \leq b, x \geq 0$ where $m < n$. Then the number of non-zero variables in a basic feasible solution is at most $m$. Similarly if the system is $Ax \leq b, x \in [0,1]^n$ then the number of non-integer variables in $y$ is at most $m$. For example in the knapsack LP we have $m = 1$ and hence in any basic feasible solution there can only be one fractional variable.

Now for the proof. We consider the system $Ax \leq b, -x \leq -\ell, x \leq u$ as a single system $Cx \leq d$ which has $m+2n$ inequalities. Since $y$ is a basic feasible solution to this system, from the definition, it is the unique solution of sub-system $C'x = d'$ where $C'$ is a $n \times n$ full-rank matrix. How many rows of $C'$ can come from $A$? At most $rank(A) \leq m$ rows. It means that the rest of the rows of $C'$ are of the from the other set of inequalities $-x \leq \ell$ or $x \leq u$. There are at least $n - rank(A)$ such rows which are tight at $y$. Thus $n - rank(A)$ variables in $y$ are tight at lower or upper bounds and hence there can only be $rank(A)$ fractional variables in $y$.

See [1] for iterated rounding based methodology for exact and approximation algorithms. The whole methodology relies on properties of basic feasible solutions to LP relaxations of combinatorial optimization problems.

## References

[1] Lap Chi Lau, R. Ravi and Mohit Singh. Iterative methods in Combinatorial Optimization. Cambridge University Press, 2011.