Fall 2013, CS 583: Approximation Algorithms

Homework 6
Due: 12/09/2013

Instructions and Policy: Each student should write up their own solutions independently. You need to indicate the names of the people you discussed a problem with; ideally you should discuss with no more than two other people.

Solve as many problems as you can. I expect at least three for this homework.

Please write clearly and concisely - clarity and brevity will be rewarded. Refer to known facts as necessary. Your job is to convince me that you know the solution, as quickly as possible.

Problem 1 Consider MAX-CUT with the additional constraint that specified pairs of vertices be on the same/opposite sides of the cut. Formally, we are given two sets of pairs of vertices, $S_1$ and $S_2$. The pairs in $S_1$ need to be separated, and those in $S_2$ need to be on the same side of the cut sought. Under these constraints, the problem is to find a maximum-weight cut.

1. Give an efficient algorithm to check if there is a feasible solution.

2. Assuming there is a feasible solution, give a strict quadratic program and vector program relaxation for this problem. Show how the algorithm for MAX-CUT we saw in class can be adapted to this problem while maintaining the same approximation ratio.

Problem 2 Given an $n \times n$ matrix $A$, a principal submatrix of $A$ is a square submatrix obtained by picking a set of indices $S \subseteq \{1, 2, \ldots, n\}$, and discarding the rows and columns of $A$ indexed by $S$. For instance, the principal submatrix of $A$ corresponding to $S = \{1, 4, 5\}$ is the $(n - 3) \times (n - 3)$ matrix obtained from $A$ by discarding rows 1, 4, 5 and columns 1, 4, 5.

Prove that all the principal submatrices of a positive semidefinite matrix are also positive semidefinite. (Which characterization of positive semidefinite matrices can you use?) Conclude that if a matrix $A$ is positive semidefinite, the determinant of any principal submatrix of $A$ is nonnegative.

(Note: The converse is also true, though you do not have to prove it: If the determinants of all principal submatrices of a real symmetric matrix $A$ are nonnegative, $A$ is positive semidefinite.)

Problem 3 In this problem you will consider the node-weighted Steiner tree problem. The input consists of an undirected graph $G = (V, E)$ and a subset $S \subseteq V$ of terminals. Each
node $v$ has a non-negative weight $w(v)$. The goal is to find a minimum weight subset of nodes $S'$ such that the subgraph induced on those nodes $G[S']$ connects all terminals. One can equivalently phrase it as finding a tree $T$ in $G$ that contains all the terminals with the goal of minimizing the weight of the nodes in $T$. It is useful to assume that the terminals have zero weight since they have to be included anyway.

- Show that an $\alpha(|S|)$-approximation for the above problem implies an $\alpha(n)$-approximation for the set cover problem on $n$ elements.

- Derive an $O(\log |S|)$-approximation as follows. A spider is a tree in which at most one node has degree more than 2. For example a path is a spider. If a spider is not a path then let $v$ be the node with degree strictly more than 2. Then the spider consists of paths $P_1, \ldots, P_k$ that are node-disjoint except at $v$. Given a spider let its density be the ratio of the weight of its nodes to the number of terminals it contains.
  
  - Let $T^*$ be a tree that contains the terminals. Then show that there is a spider in $T^*$ of density at most $w(T^*)/|S|$.
  
  - Show how one can compute a minimum density spider in polynomial time.
  
  - Combine the above two to derive the $O(\log n)$-approximation.

**Problem 4** Problem 6.6 from the Williamson-Shmoys book.

**Problem 5** Problem 15.4 from the Williamson-Shmoys book.