Fall 2013, CS 583: Approximation Algorithms

Homework 6
Due: 12/09/2013

Instructions and Policy: Each student should write up their own solutions independently. You need to indicate the names of the people you discussed a problem with; ideally you should discuss with no more than two other people.

Solve as many problems as you can. I expect at least three for this homework.

Please write clearly and concisely - clarity and brevity will be rewarded. Refer to known facts as necessary. Your job is to convince me that you know the solution, as quickly as possible.

Problem 1 Consider MAX-CUT with the additional constraint that specified pairs of vertices be on the same/opposite sides of the cut. Formally, we are given two sets of pairs of vertices, S_1 and S_2. The pairs in S_1 need to be separated, and those in S_2 need to be on the same side of the cut sought. Under these constraints, the problem is to find a maximum-weight cut.

1. Give an efficient algorithm to check if there is a *feasible* solution.
2. Assuming there is a feasible solution, give a strict quadratic program and vector program relaxation for this problem. Show how the algorithm for MAX-CUT we saw in class can be adapted to this problem while maintaining the same approximation ratio.

Problem 2 Given an $n \times n$ matrix A, a *principal submatrix* of A is a square submatrix obtained by picking a set of indices $S \subseteq \{1, 2, \ldots, n\}$, and discarding the rows *and* columns of A indexed by S. For instance, the principal submatrix of A corresponding to $S = \{1, 4, 5\}$ is the $(n-3) \times (n-3)$ matrix obtained from A by discarding rows $1, 4, 5$ and columns $1, 4, 5$.

Prove that *all* the principal submatrices of a positive semidefinite matrix are also positive semidefinite. (Which characterization of positive semidefinite matrices can you use?) Conclude that if a matrix A is positive semidefinite, the determinant of any principal submatrix of A is nonnegative.

(*Note:* The converse is also true, though you do not have to prove it: If the determinants of all principal submatrices of a real symmetric matrix A are nonnegative, A is positive semidefinite.)

Problem 3 In this problem you will consider the *node-weighted* Steiner tree problem. The input consists of an undirected graph $G = (V, E)$ and a subset $S \subseteq V$ of terminals. Each node v has a non-negative weight $w(v)$. The goal is to find a minimum weight subset of nodes S' such that the subgraph induced on those nodes $G[S']$ connects all terminals. One can equivalently phrase it as finding a tree T in G that contains all the terminals with the goal of minimizing the weight of the nodes in T. It is useful to assume that the terminals have zero weight since they have to be included anyway.

- Show that an $\alpha(|S|)$-approximation for the above problem implies an $\alpha(n)$-approximation for the set cover problem on n elements.
- Derive an $O(\log |S|)$-approximation as follows. A *spider* is a tree in which at most one node has degree more than 2. For example a path is a spider. If a spider is not a path then let v be the node with degree strictly more than 2. Then the spider consists of paths $P_1, \ldots, P_k$ that are node-disjoint except at v. Given a spider let its density be the ratio of the weight of its nodes to the number of terminals it contains.
  - Let T^* be a tree that contains the terminals. Then show that there is a spider in T^* of density at most $w(T^*)/|S|$.
  - Show how one can compute a minimum density spider in polynomial time.
  - Combine the above two to derive the $O(\log n)$-approximation.

Problem 4 Problem 6.6 from the Williamson-Shmoys book.

Problem 5 Problem 15.4 from the Williamson-Shmoys book.