

## Fall 2013, CS 583: Approximation Algorithms

### Homework 1

Due: 09/20/2013

**Instructions and Policy:** Each student should write up their own solutions independently. You need to indicate the names of the people you discussed a problem with; ideally you should discuss with no more than two other people.

Solve as many problems as you can. Please submit your solutions to at least 4 problems.

Please write clearly and concisely - clarity and brevity will be rewarded. Refer to known facts as necessary. Your job is to convince me that you know the solution, as quickly as possible.

**Problem 1** Consider the  $k$ -dimensional knapsack problem. We are given  $n$  non-negative  $k$ -dimensional vectors  $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n$ . Each vector has a non-negative weight,  $w_i$  for  $\bar{v}_i$ . We are also given a  $k$ -dimensional knapsack  $\bar{V}$  and the goal is to find a maximum weight subset of vectors that pack in to  $\bar{V}$ . A subset of vectors pack into  $\bar{V}$  if their vector addition is less than  $\bar{V}$  (co-ordinate wise). Prove that there is a pseudo-polynomial time algorithm for this problem when  $k$  is a fixed constant independent of  $n$ . What is the running time of your algorithm? Prove that even for  $k = 2$  and *unit weights* the problem does not admit an FPTAS. (Hint: use a reduction from the Partition problem.) **Extra credit:** Obtain a PTAS for this problem when  $k$  is a fixed constant independent of  $n$  (might need to use LP techniques).

**Problem 2** Consider a budgeted version of the maximum coverage problem. We are given  $m$  sets  $S_1, S_2, \dots, S_m$ , each a subset of a set  $\mathcal{U}$ . Each set  $S_i$  has a non-negative cost  $c_i$  and we are also given a budget  $B$ . The goal is to pick sets of total cost at most  $B$  so as to maximize the number of elements covered. Show that if  $c_i \leq \epsilon B$  for  $1 \leq i \leq m$  then the Greedy algorithm yields a  $1 - 1/e - \epsilon$  approximation (for sufficiently small  $\epsilon$ ). For any fixed  $\epsilon > 0$  obtain a  $1 - 1/e - \epsilon$  approximation. (Hint: consider the PTAS for knapsack from the lectures. You might find the inequality  $1 - x \leq e^{-x}$  useful.) **Extra credit:** Obtain a  $1 - 1/e$  approximation for this problem.

**Problem 3** In the submodular set cover problem, we are given a universe  $\mathcal{U}$  of elements, and a monotone submodular function  $f: 2^{\mathcal{U}} \rightarrow \mathcal{R}^+$ . The goal is to pick a smallest  $U' \subseteq \mathcal{U}$  such that  $f(U') = f(\mathcal{U})$ . In the submodular maximum coverage problem, we are also given an integer  $k$ , and the goal is to pick a set  $U' \subseteq \mathcal{U}$  of size at most  $k$  to maximize  $f(U')$ .

1. Prove that the greedy algorithm gives a  $1 - 1/e$  approximation for submodular maximum coverage.

2. Consider a  $k \times \ell$  matrix  $M$  where each entry  $M_{i,j}$  is a subset of a universe  $\mathcal{U}$  of size  $n \geq \ell$ . We say a row  $M_i$  of  $M$  is *covered* by  $U' \subseteq \mathcal{U}$  if there are  $\ell$  *distinct* elements  $e_1, e_2, \dots, e_\ell$  in  $U'$  such that  $e_j \in M_{i,j}$  for each  $1 \leq j \leq \ell$ .

Prove that the problem of finding a smallest  $U' \subseteq \mathcal{U}$  such that each row is covered by  $U'$  is a special case of the submodular set cover problem.

**Problem 4** Problem 3.6 from Shmoys-Williamson book.

**Problem 5** Problem 13.4 from Vazirani book.

**Problem 6** Problem 1.4 from Shmoys-Williamson book.