

CS 580: Algorithmic Game Theory, Fall 2022

HW 2 (due on Thursday, 6th October at 11:59pm CST)

Instructions:

1. We will grade this assignment out of a total of 40 points.
2. You can work on any homework in groups of (\leq) two. Submit only one assignment per group. First submit your solutions on Gradescope and you can add your group member after submission.
3. If you discuss a problem with another group then write the names of the other group's members at the beginning of the answer for that problem.
4. Please type your solutions if possible in Latex or doc whichever is suitable, and submit on Gradescope.
5. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.
6. Except where otherwise noted, you may refer to lecture slides/notes. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you **cite them appropriately**, and make sure to **write in your own words**.
7. No late assignments will be accepted.
8. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available at Prof. Vijay V. Vazirani's webpage.

1. (DSIC Auctions)

- (a) Consider the following allocation rules for a single-parameter environment auction. For each of these, prove if they are implementable (monotone) or not.
 - i. (2.5 points) The allocation rule that maximizes the social welfare among all feasible allocations. That is, for bids b_1, b_2, \dots, b_n , $x(b) = \operatorname{argmax}_x \sum_i b_i x_i$.
 - ii. (2.5 points) Suppose the set of feasible allocations is $X = \{x \in [0, 1]^n \mid \sum_i x_i = 1\}$. Give each agent a fraction of the item proportional to the distance of their bid from the average of all bids (normalized to 1), and additionally have $x_i = 0$ when $b_i = 0$. That is, if b_1, b_2, \dots, b_n are the bids of all agents, and if we let $b_{avg} = \sum_i b_i / n$, then the allocation $x(b)$ satisfies for all agents i, j where $x_i, x_j > 0$, $\frac{x_i}{x_j} = \frac{|b_i - b_{avg}|}{|b_j - b_{avg}|}$, and $\sum_i x_i = 1$.

- (b) (5 points) We want to auction m heterogeneous items among n unit demand bidders, where bidder i has value v_{ij} for item j , and her value for set $S \subset \{1, \dots, m\}$ is $\max_{j \in S} v_{ij}$. Design a polynomial-time algorithm to implement the VCG auction.
2. (Voting) As discussed in the class, the *Borda count* constructs a social ranking out of individual rankings as follows: Suppose that there are k candidates. Each individual submits a ranking of the candidates from 1st to k th (for the purposes of this problem, you can assume that the ranking is strict; there are no ties.). This translates to giving $(k - 1)$ "points" to the first candidate, $(k - 2)$ to the second candidate, and in general $(k - i)$ points to the i th candidate. The final order ranks candidate in the decreasing order of their total points.
- (a) (2 points) Which of the assumptions in Arrow's theorem are violated?
- (b) (3 points) Give an example to demonstrate your claim.
- Assume now we run the following voting scheme: We create a directed graph G with a vertex for every candidate and an edge (u, v) if and only if total points of u is strictly more than the total points of v . Then, we construct the *strongly connected component* meta-graph G_{SCC} of G and return as a winner a vertex at random from the source meta-vertex of G_{SCC} (recall that G_{SCC} is necessarily a DAG).
- (c) (2 points) Which of the assumptions in Arrow's theorem are violated by this voting scheme?
- (d) (3 points) Give an example to demonstrate your claim.
Hint: Can G have cycles?

3. (Stable Matching) In practice, it may be unrealistic to expect that each person can provide a strict preference ordering of all their potential mates. To reflect this, let's suppose we allow each person's preference list to contain ties. Mathematically, a set W of k women forms a tie of length k in the preference list of man m if m does not prefer w_i to w_j for any $w_i, w_j \in W$ (i.e., m is indifferent between w_i and w_j), while for any other woman $w \notin W$, either m prefers w to all women in W or m prefers all the women in W to w . A tie on a woman's list is defined analogously. We will call this the *Partial Order Stable Marriage Problem*. Recall that in the original Stable Marriage Problem, where preference lists are strictly ordered, it is always possible to find a stable matching, where a matching is stable if there is no man x and woman y such that x and y both prefer each other over their current partners. However, for the Partial Order Stable Marriage Problem there exist three different types of stability:
- **Weak stability.** A matching is weakly stable if there is no couple x and y , each of whom strictly prefers the other to their current partner in the matching.
 - **Strong stability.** A matching is strongly stable if there is no couple x and y such that x strictly prefers y to his or her partner, and y either strictly prefers x to his or her partner or is indifferent between them.
 - **Super-stability.** A matching is super-stable if there is no couple x and y , each of whom either strictly prefers the other to his/her partner or is indifferent between them.

It is then natural to ask whether these different types of stable matchings always exist in a given instance of the Partial Order Stable Marriage Problem.

- (a) (3 points) For the Partial Order Stable Marriage Problem, does a weakly stable matching always exist? Either prove the statement or provide a counterexample.
- (b) (2 points) For the Partial Order Stable Marriage Problem, does a strongly stable matching always exist? Either prove the statement or provide a counterexample.
- (c) (1 points) For the Partial Order Stable Marriage Problem, does a super-stable matching always exist? Either prove the statement or provide a counterexample.
- (d) (4 points) Assume we are given an instance of the Partial Order Stable Marriage Problem, along with a weakly stable matching M for that instance. Upon getting married to their partners assigned in M , each person's preferences change slightly and the married partner becomes preferred over anyone they were tied with, but doesn't change in rank otherwise. Is M now super-stable? If so, prove this formally, else provide a counterexample.

4. (Revenue and Welfare)

- (a) (5 points) Consider the following problem of designing a *posted price mechanism*, which relates to prophet inequalities. There are n buyers for a single item and we want to sell the item to one of them. Each buyer i arrives at our shop with probability p_i , and they don't arrive at all with probability $1 - p_i$. If buyer i arrives, their value for the item is v_i . The buyers choose to arrive or not independently of each other, and we know that $\sum_{i=1}^n p_i = 1$ (on expectation, one buyer will arrive).

Consider the following mechanism: When buyer i arrives, if we haven't sold the item yet, we sell it to i with probability $1/2$, else we pass with probability $1/2$. Show that the expected value of this mechanism is a $1/4$ -approximation to the expected maximum value $\max_i p_i v_i$.

Hint: Recall the proof of the prophet inequality; notice that

$$\mathbb{E}[ALG] = \sum_{i=1}^n \Pr[\text{Item is sold to buyer } i] \cdot v_i.$$

Use this, along with Markov's inequality, to upper bound the probability the item is not sold at all. Markov's inequality says that for a non-negative random variable X and value $a > 0$, we have $\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$.

- (b) (5 points) This problem considers a variation of the Bulow-Klemperer theorem. Consider selling $k \geq 1$ identical items (with at most one given to each bidder) to bidders with valuations drawn iid from F , where F is a regular distribution (i.e., the corresponding ϕ^{-1} is a monotonically increasing function). Prove that for every $n \geq k$, the expected revenue of the Vickrey auction (with no reserve) with $n + k$ bidders is at least that of the Myerson's optimal auction for F with n bidders.

Hint: Myerson's optimal auction will be Vickrey with reserve $\phi^{-1}(0)$, i.e., discard bids below $\phi^{-1}(0)$, give the item to k highest bidders and charge them $\max(k + 1)^{\text{th}}$ highest bid, $\phi^{-1}(0)$

5. (Bonus questions)

- (a) Consider the reverse auction we briefly talked about in class: A denotes the set of bidders who are willing to sell the spectrum they hold. We say that set $S \subset A$ is feasible, if we

can repack $A \setminus S$ in the available range given that S is acquired. Clearly, the set $F \subseteq 2^B$ of feasible sets is upward closed (i.e., supersets of feasible sets are again feasible).

- Initialize $S = A$.
- While there is a bidder $i \in S$ such that $S \setminus \{i\}$ is feasible:
 - (*) Delete some such bidder i from S such that $S \setminus \{i\}$ is feasible.
- Return S .

Suppose we implement the step (*) using a scoring rule, which assigns a number to each bidder i . At each iteration, the bidder with the largest score (whose deletion does not destroy feasibility of S) gets deleted. The score assigned to a bidder i can depend on i 's bid, the bids of other bidders that have already been deleted, the feasible set F , and the history of what happened in previous iterations. (Note a score is not allowed to depend on the value of the bids of other bidders that have not yet been deleted.) Assume that the scoring rule is increasing – holding everything fixed except for b_i , i 's score is increasing in its bid b_i . Then, show that the allocation rule above is monotone: for every i and b_{-i} , if i wins with bid b_i then she will keep winning with any bid less than b_i .

- (b) For the same setting as question 4.(a), suppose now that the buyers arrive in random order, which you can model as having each buyer decide on a time $t \in [0, 1]$ to arrive, uniformly at random and independently from the other buyers. Consider the following mechanism: When buyer i arrives at time t , if we haven't sold the item yet, we sell it to i with probability $e^{-t \cdot p_i}$, else we pass with probability $1 - e^{-t \cdot p_i}$. Show that the expected value of this mechanism is a $(1 - 1/e)$ -approximation to the expected maximum value $\max_i p_i v_i$.

Hint: Show this mechanism ensures that we sell the item to a buyer that arrives to our shop with probability at least $1 - 1/e$. Again, upper bound the probability the item is not sold at all. For a fixed time t , calculate the probability we sell the item to i at time t and integrate over $t \in [0, 1]$. You might find useful the fact that $\int_0^t e^{-p_i \cdot u} du = \frac{1 - e^{-p_i \cdot t}}{p_i}$.

- (c) **[Difficult]** We mentioned in class that one can design a posted-price mechanism for a single-item auction using the prophet inequality, and obtain guarantees about the approximation to the optimal revenue of the auction via the use of virtual valuations and Myerson's optimal auction. In this problem, we will show that this can be done in *procurement auctions* as well, where one wants to design a posted-price mechanism for a single-item procurement auction and wants to approximate the optimal (minimum) cost (price) paid.

Specifically, assume that there exists a single buyer and n sellers. The buyer wants to procure a single item from the sellers, and each seller's price offer to the buyer is drawn independently from the same known distribution D . The buyer wants to design a price threshold such that if they observe the offers in any order, then accepting the first price that is below the price threshold will yield a good enough approximation to the minimum cost paid.

Before we continue, consider the following setting: Suppose you have n random variables X_1, \dots, X_n drawn independently from the same known distribution D . At step i , you observe the realization of X_i . The goal is to select the minimum possible value and you compare against the optimal (minimum) value if you could see all the realizations at

once, i.e. $\mathbb{E}[\min_i X_i]$. A *cost prophet inequality* is an algorithm that selects a threshold T and accepts the first realization that is below T . We say that a cost prophet inequality guarantees an α -approximation to $\mathbb{E}[\min_i X_i]$ (where $\alpha \geq 1$), if it selects a cost C and $\mathbb{E}[C] \leq \alpha \cdot \mathbb{E}[\min_i X_i]$.

Show that if there exists a cost prophet inequality that guarantees an α -approximation to $\mathbb{E}[\min_i X_i]$, then one can design a price threshold that guarantees that, on expectation, the buyer will not pay more than $\alpha \cdot P^*$ where P^* is the minimum price offered.

Hint: Use the same proof as Myerson's optimal auction but instead of virtual valuations, use the virtual costs: $\phi(c) = c + \frac{F(c)}{f(c)}$.