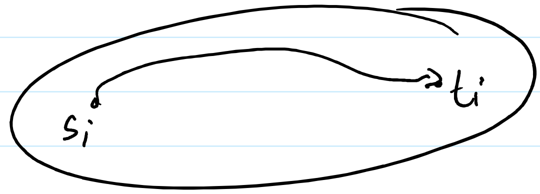
Games w/ Positive Externalities

Tuesday, November 9, 2021 1:56 PM

★ n/w Costsharing Game / n/w formation Game.

→ $G = (N, E)$ directed.

→ $N = \{1, \dots, n\}$ players.



→ $i \in N$, $s_i, t_i \in V$, wants to "build" path from s_i to t_i

→ cost of building edge e is r_e.

Strategy set of player i $\mathcal{P}_i$ = $s_i - t_i$ paths.

$\forall i: P_i \in \mathcal{P}_i$, $P = (P_1, \dots, P_n)$ is a strategy profile.

$\forall e \in E$ $\quad f_e$ = # players who has $e \in P_i$.

$= |\underbrace{\{i \in N \mid e \in P_i\}}|$

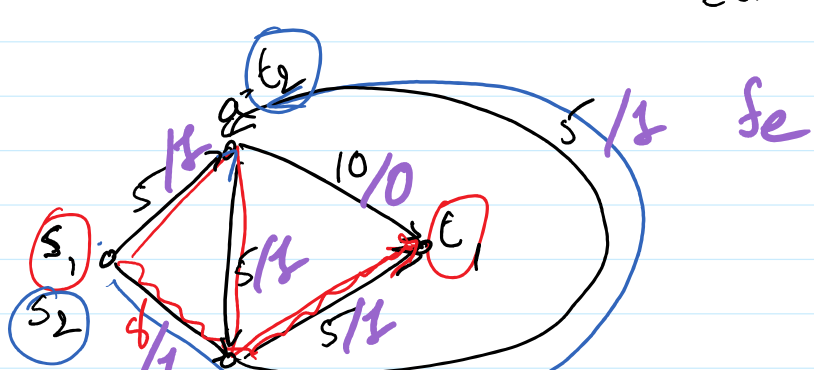
share cost of $\frac{r_e}{f_e}$ for building edge e.

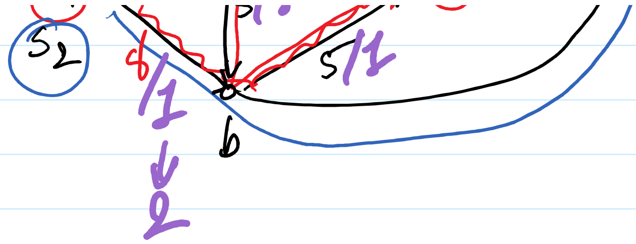
$$C_i(P) = \sum_{e \in P_i} \frac{r_e}{f_e}$$

$$\mathrm{Cost}(P) = \sum_{i=1}^{n} C_i(P) = \sum_{i=1}^{n} \sum_{e \in P_i} \frac{r_e}{f_e} = \sum_{e \in E} \frac{r_e}{\cancel{f_e}} \left(\sum_{i : e \in P_i} 1 \right) = \cancel{f_e}$$

$$= \sum_{e \in E : f_e \geq 1} r_e$$

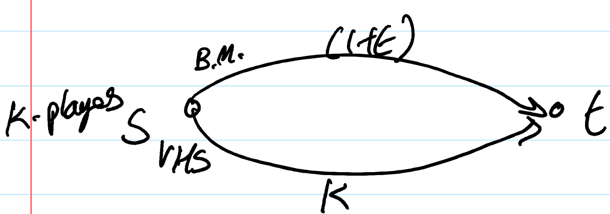
★ Example



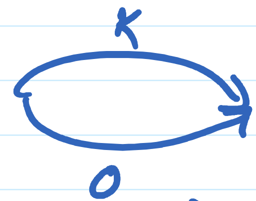


★ VHS vs Betamax (1980's)

Earlier — later but far better technology.



OPT =



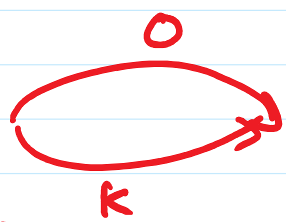
→ P.P. cost is $\frac{1+\varepsilon}{K}$

⇓

NE 1

cost(OPT) = C(1+ε)

NE 2 :



P.P. cost = 1

cost(NE 2) = K

$$PoA \geq \frac{\text{cost}(NE\ 2)}{\text{cost}(OPT)}$$

$$= \frac{K}{1+\varepsilon} \sim K = \#\text{ players.}$$

★ Suppose we "select" good NE by forcing coordination.

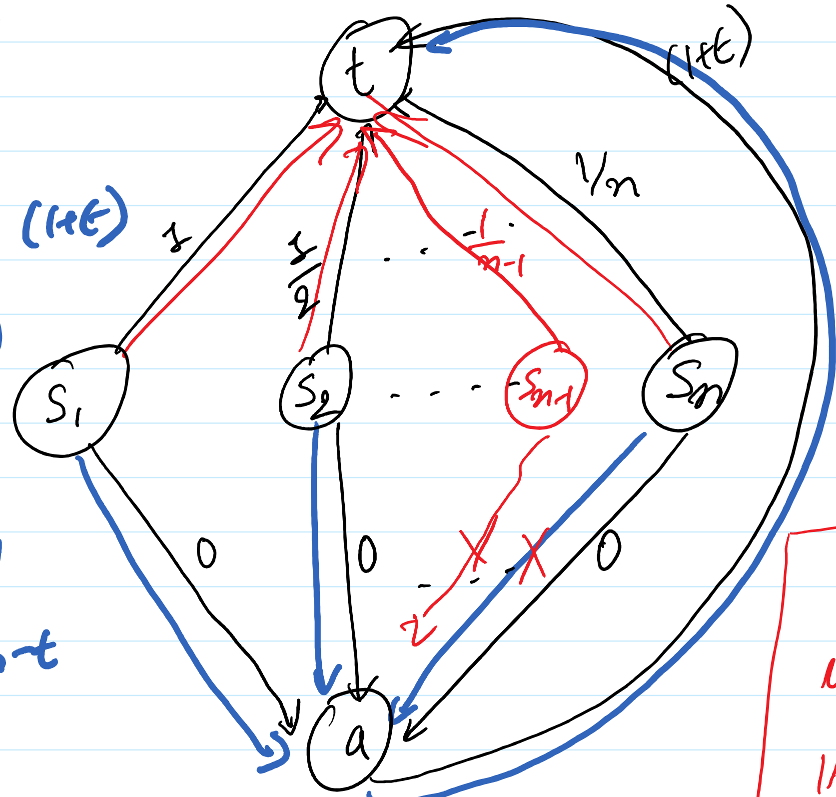
Mediator / default option.

$$\text{Price-of-Stability} = \frac{\text{Cost of the "best" NE}}{\text{cost of OPT}}$$

$$\left(1 \le PoS \le PoA\right) \qquad = \frac{\min_{\substack{P \in \mathcal{P} \\ P \text{ is a NE}}} \text{Cost}(P)}{\text{Cost}(OPT) = \min_{P} \text{Cost}(P)}$$

Q: How bad can PoS be?

Eg 2: n-player



OPT

$\text{cost}(OPT) = (1+\epsilon)$

$\forall i \quad c_i(OPT) = \frac{(1+\epsilon)}{n}$

$\Downarrow$

n^{th} player will deviate to $s_n \to t$

NE (The only NE)

$\text{cost}(NE) = 1 + \frac{1}{2} + \frac{1}{3} \dots + \frac{1}{n}$

$= H_n \sim \ln n$

$A \subseteq N$

$i^* = \arg\max_{i \in A} A$

$|A| \le i^*$

$\forall i \in A \quad c_i(P^A) \ge \frac{(1+\epsilon)}{i^*}$

$\Rightarrow i^*$ can deviate to $s_{i^*} \to t$ & pay $\frac{1}{i^*}$

$PoS = \frac{H_n}{1+\epsilon} \sim H_n.$

Thm: $PoS \le H_n$ for n-player n/w cost sharing games.

Q: why there always exist pure NE?

Ans $\because$ it is a potential game.

☆ Potential Func. $\left(\sum_{e \in E} \sum_{k=1}^{f_e} c_e(k) \text{ in routing game} \right)$

$\parallel$ (in C.S. game)

$\frac{r_e}{k}$

$$\phi(P) = \sum_{e \in E} r_e \sum_{k=1}^{f_e} \frac{1}{k}$$

Prop: $\forall P \in \mathcal{P}, \ \forall i, \ P_i' \in \mathcal{P}_i$

$$C_i(P_i', P_{-i}) - C_i(P_i, P_{-i}) = \phi(P_i', P_{-i}) - \phi(P_i, P_{-i})$$

Proof: Exe.

Proof of Thm: $PoS \le H_n$

$P \in \mathcal{P}$

$$\text{Cost}(P) = \sum_{i=1}^{n} C_i(P) = \sum_{e \in E : f_e \ge 1} r_e \le \overbrace{\sum_{e \in E} r_e \sum_{k=1}^{f_e} \frac{1}{k}}^{\phi(P) =}$$

$$(\because f_e \le n) \le \sum_{e \in E : f_e \ge 1} r_e \sum_{k=1}^{n} \frac{1}{k} = H_n$$

$$= H_n \left(\sum_{e : f_e \ge 1} r_e \right)$$

$$= H_n \cdot cost(P)$$

$$\Downarrow$$

$$\boxed{cost(P) \le \phi(P) \le H_n \cdot cost(P)} \longrightarrow (1)$$

$$\boxed{P' = \arg\min_{P} \phi(P)} \Rightarrow P' \text{ is a NE.}$$

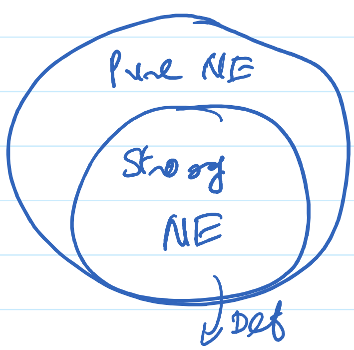
P^*: OPT

$$PoS = \frac{\min_{P:\, P \text{ is NE}} cost(P)}{cost(P^*)} \le \frac{cost(P')}{cost(P^*)}$$

$$cost(P') \underset{(\because 1)}{\le} \phi(P') \le \phi(P^*) \underset{(\because 1)}{\le} H_n \, cost(P^*)$$

$$\Rightarrow PoS \le \frac{cost(P')}{cost(P^*)} \le H_n. \quad \blacksquare$$

☆ Strong NE



$$PoS^{strongNE} \ge PoS^{PNE}$$

$$PoA^{strongNE} \le PoA^{strongNE}$$

There is NO "Beneficial coaliation".

→ Beneficial coaliation: w.r.t. $P = (P_1, \ldots, P_n)$

$A \subseteq N$ s.t. $\exists\, P'_A \in \times_{i \in A} \mathcal{P}_i$

$\forall i \in A: \quad C_i(P'_A, P_{-A}) \leq C_i(P_A, P_{-A})$

& strict inequality for atleast one player $i \in A$.

Thm: $PoA^{\text{Strong NE}} \leq H_n \sim \ln n$.

Pf: P: a strong NE $\qquad P^*$: OPT.

Q_n: $A_n = \{1, \dots, n\}$ is not beneficial!

$\because \exists\, i \in A_n$ s.t. $C_i(P^*) > C_i(P)$. Denote i as n

$Q_{(n-1)}$: $A_{n-1} = \{1, \dots, n-1\}$ is not beneficial?

$\because \exists\, i \in A_{n-1}$ s.t. $C_i(P^*_{A_{n-1}}, P_n) > C_i(P)$. Denote i as $(n-1)$

$\vdots$

Q_K $\quad A_K = \{1, \dots, K\}$ is not beneficial?

$\exists\, i \in A_K$ s.t. $C_i(P^*_{A_K}, P_{-A_K}) > C_i(P)$. Denote i as K.

$$\text{cost}(P) = \sum_{K=1}^{n} C_K(P) < \sum_{K=1}^{n} C_K(P^*_{A_K}, P_{-A_K})$$

$$\leq \sum_{K=1}^{n} C_K(P^*_{A_K}) \quad \longrightarrow \text{(1)}$$

$\hookrightarrow$ Remove $N \setminus A_K = -A_K$ from the system &

A_k plays $P^*_{A_k}$.

$\star$ $P^*_{A_k}$

f_e^k = # players in A_k building edge k in $P^*_{A_k}$.

$$\boxed{c_k(P^*_{A_k}) = \sum_{e \in P^*_i} \frac{r_e}{f_e^k} \overset{?}{=} \underline{\phi(P^*_{A_k})} - \phi(P^*_{A_{k-1}})}$$

Exe.

$$\sum_{k=1}^{n} c_k(P^*_{A_k}) = \sum_{k=1}^{n} \phi(P^*_{A_k}) - \phi(P^*_{A_{k-1}})$$

$$= \phi(P^*_{A_n}) - \cancel{\phi(P^*_{A_{n-1}})}$$

$$+ \cancel{\phi(P^*_{A_{n-1}})} - \cancel{\phi(P^*_{A_{n-2}})}$$

$$\vdots$$

$$+ \cancel{\phi(P^*_{A_1})} - \underset{\overset{\shortparallel}{0}}{\phi(P^*_{A_0})}$$

$$= \underset{\overset{\shortparallel}{P^*}}{\phi(P^*_{A_n})} \quad (\because A_n = \{1, \ldots, n\})$$

$$= \phi(P^*)$$

$\therefore$ (1) $\quad \text{cost}(P) \leq \phi(P^*) \leq H_n \, \text{cost}(P^*)$

$$\text{PoA} = \frac{\text{cost}(P)}{\text{cost}(P^*)} \leq H_n$$

∎