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2019-01-24.4 → 2019-01-29.1
2019-01-29.2 CS579

CS579: Computational Complexity: Lecture 5

22 25

admin: - ps 1 due 01-31 @ 3:30pm [email to course staff] [piazza annc]
- Zanders office hours - cancelled tomorrow [weather] [rescheduled]
- rescheduled to today 6-8pm SC3303

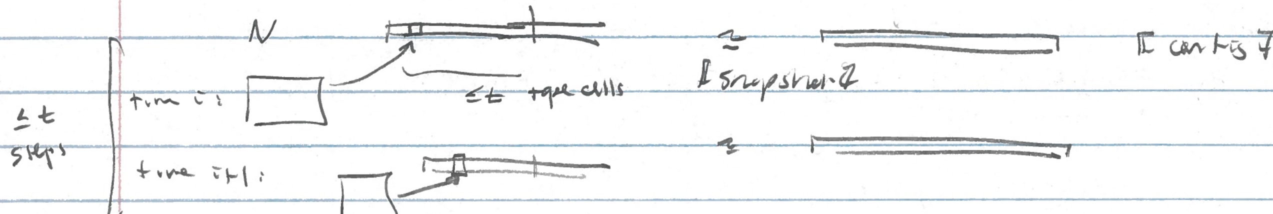
last time: - reducibility
- NP completeness
- 3SAT ≤ CLIQUE

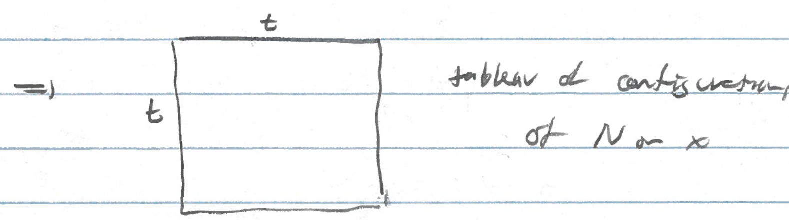
today - Cook Levin Thm:

thm [Cook 71, Levin 73]: 3SAT is NP-complete
- 3SAT ∈ NP [easy]
- ∀ A ∈ NP $A \le_p$ 3SAT [NP-hard] [today]

plan: $A \le_p$ SAT [most of ideas]
A ≤ 3SAT [by modification]

idea: A in NP, decided by NTM N time $t(n) \in poly(n)$ [such NTM exists]
$f: x \in \Sigma^* \longmapsto \varphi_x$
$x \in A$ iff $\varphi_x \in SAT$
↳ will simulate N on x
"says" "some branch of N accepts x"
vars y of φ_x encode branch





def: NTM N - states Q
- tape alphabet Γ [assume disjoint]

a **configuration** is a string over $C := Q \sqcup \Gamma \sqcup \#$ ← end markers (Γ contains '␣')
- '#' appears only in first/last position
- exactly one symbol of Q appears
← - state of NTM
- position of head to the right

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ex NTM N on x has starting configuration # q_{start} $x_1 \cdots x_n$ ⊔ ⋯ ⊔ # (t − n)

⟹ TM uses t cells ⟹ config has t+3 symbols [head = 1, boundary = 2]

def. for NTM N on x in time t, a tableau is a sequence of t configurations of lengths t+3 ⟺ t × (t+3) matrix

an accepting tableau has t configs
- row i = config of ith step of N on x
  - row 1 = initialization
  - row i non-deterministically follows from row i−1
- last row is accepting config ← has q_{acc}

Q: if N halts early?
A: configs freeze on halting

Clm: NTM N accepts x in ≤ t steps iff ∃ accepting tableau size t × (t+3)
Pf: by defn
time ≤ t ⟺ ≤ t rows
≤ t ⟹ ≤ t+3 cols

Q. what is t(n)?
[t] might be complicated, need to construct it?
Lem: [technical lem]

$\exists c, t(n) \le O(n^c) \Rightarrow \exists d$ st $\forall n$ $t(n) \le n^d + d$

Pf: $\Rightarrow \exists n_0, c_0 \in \mathbb{N}$ $\forall n \ge n_0$ $t(n) \le c_0 n^c$

$n_1 := \max\{n_0, c_0\}$

$\Rightarrow \exists n_1$ $\forall n \ge n_1$ $t(n) \le c_0 n^c \le n_1 \cdot n^c \le n^{c+1}$

$c_1 := \max_{n \le n_1} t(n) \Rightarrow \forall n \le n_1$ $t(n) \le c_1 \le n^{c+1} + c_1$

$\exists n_1$ $\le n^{c+1} \le n^{c+1} + c_1$

$d = \max\{c+1, c_1\}$ $\forall n$ $\le n^d + d$

Pf of Cook-Levin. $A \in NP$ accepted by NTM N – runs in time $\le t(n) = n^c + c$
- states Q
- tape Γ

plan. NTM N acc x iff NTM N acc x in t(n) steps
iff " , via some branch
iff exists t × (t+3) size accepting tableau
iff φ_x has satisfying assignment ⟺ says yes
(need to construct) [depends on x]

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φ : vars $y_{i,j,a}$ $a \in [0,1]$ $1 \le i,j \le t$, $a \in Q \cup \Gamma \cup \{\#\}$
want $= 1$ iff cell$[i,j] = a$ in tableau $=: C$

$\varphi := \varphi_{cell} \wedge \varphi_{start} \wedge \varphi_{accept} \wedge \varphi_{move}$
[each cell well defined] [start correctly] [accept] [transition] [most interesting]

$$\varphi_{cell} = \bigwedge_{1 \le i,j \le t} \left(\left(\bigvee_{a \in C} y_{ija} \right) \wedge \bigwedge_{a \ne b \in C} \left(\overline{y_{ija}} \vee \overline{y_{ijb}} \right) \right)$$

$a \in C$: ≥ 1 symbol chosen; $a \ne b \in C$: ≤ 1 symbol chosen
$\Rightarrow = 1$ symbol per cell

note: is CNF if $C_1 \wedge \cdots \wedge C_K$ $(\ell_1 \vee \cdots \vee \ell_{n_k})$

$\varphi_{start} := y_{11\#} \wedge y_{12 q_0} \wedge y_{13 x_1} \wedge y_{14 x_2} \wedge \cdots \wedge y_{1(n+2) x_n} \wedge y_{1(n+3)\sqcup} \cdots y_{1(t-2)\sqcup} y_{1(t-1)\#}$

note: is CNF

$\varphi_{accept} := \bigvee_{1 \le j \le t+3} y_{tj q_{acc}}$

note is CNF

$\varphi_{move} :=$ each row follows legally from previous row
$=$ # stays #
- if cell is not overwritten by head, stays same
- if " is " , is properly overwritten
- if head moves, does correctly

$=$ every 2×3 window of tableau is valid

[continuous]

eg: $\begin{array}{|c|c|c|}\hline q & a & b \\ \hline c & q' & b \\ \hline \end{array}$ $\begin{array}{|c|c|c|}\hline 1 & b & c \\ \hline a & b & c \\ \hline \end{array}$

eg not: $\begin{array}{|c|c|c|}\hline a & b & c \\ \hline a & b' & c \\ \hline \end{array}$ [no head]

$$\varphi_{move} = \bigwedge_{\substack{1 \le i \le t-1 \\ 1 \le j \le t+1}} \underbrace{\text{2x3 window starting at } (i,j) \text{ is valid}}_{= \bigvee_{\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline \end{array} \text{ legal}} (x_{ija} \wedge x_{i+1,j,b} \wedge x_{i+2,j,c} \wedge \cdots \wedge x_{i+1,j+2,f})}$$

Rmk. is not a CNF, but can be made into one by distributive law

$$a \vee (b \wedge c) = (a \vee b) \wedge (b \vee c)$$

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the reduction: on input x output φ_x

runtime: $|\varphi_x| = |\varphi_{cell}| + |\varphi_{start}| + |\varphi_{acc}| + |\varphi_{move}|$

can construct as highly explicit: $\Theta(t^2 \cdot \lg t)$ [t^2 cells, $\lg t$ to index], $\Theta(t \cdot \lg t)$, $\Theta(t \lg t)$, $\Theta(t^2 \cdot \lg t)$ [$\Theta(1)$ per window]

correctness:

Clm: top row valid config & every 2x3 window of tableau legal
⇒ all rows valid

Pf: induction: head moves ≤ 1 spot

[no head] [head]

⇒ N acc x iff some branch accepts iff ∃ accepting tableau iff φ_x satisfiable. □

Cor: CNF-SAT is NP-complete

Cor: 3SAT NP-complete

Pf: CNF-SAT ≤ 3SAT

$C_1 \wedge \dots \wedge C_k$ → $C_1' \wedge \dots \wedge C_{k'}'$

$l_1 \vee \dots \vee l_m$, $l_1' \vee l_2' \vee l_3'$

reduction: $l_1 \vee l_2 \vee l_3 \vee l_4 \vee l_5 \mapsto (l_1 \vee l_2 \vee z) \wedge (\bar{z} \vee l_3 \vee y) \wedge (\bar{y} \vee l_4 \vee l_5)$

correctness: SAT ⇒ SAT

1   1 0 1 0 1

1   1 0 1 0 1

⇐ [as on ps1]

runtime: clear

next time: space complexity