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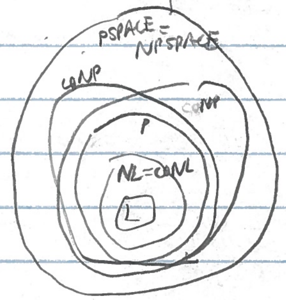
2019-02-12.4 → 2019-02-14.1
2019-02-14.2
CS579

CS579 Computational Complexity : Lecture 10 1720

<u>admin</u>: ps 2 due today
ps3 out tonight
Zander lecturing 02-21 ④

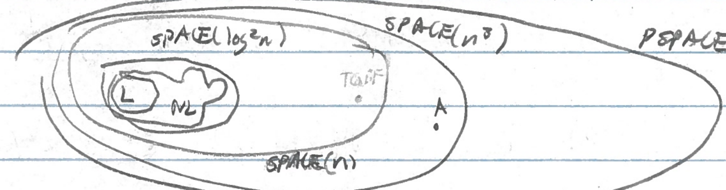
<u>last time</u>: - PATH NL-complete
- NL = coNL

<u>today</u>: hierarchy theorems



Q: L vs P? P vs NP? P vs PSPACE? L vs PSPACE?
A: ? ? ? L ⊆ NL ⊊ PSPACE

<u>space</u>:



$A \in SPACE(n^3) \setminus SPACE(n)$ [also for time]

<u>thm</u> (Space hierarchy theorem) 1965 : $s(n): \mathbb{N} \to \mathbb{N}$ w/
$s(n)$ (weakly) space constructible: $1^n \to 1^{s(n)}$ can be computed in $O(s(n))$ space
then exist language A st: $A \in SPACE(s(n))$
$A \notin SPACE(o(s(n)))$

<u>idea</u>: mimic existence of undecidable languages via diagonalization

<u>thm</u>: exist languages A st any TM M, $A \neq L(M)$ [eg: halting problem]

<u>Pf</u>. $M_1, \ldots M_i, \ldots$ enumerate all TM's [= finite strings]
Q: does M_i accept $\langle M_i \rangle$?

	$\langle M_1\rangle$	$\langle M_2\rangle$	---	$\langle M_i\rangle$
M_1	acc	acc	--	acc
M_2	rej	acc		rej
⋮				
M_i	rej	rej	--	rej

$A := \{\langle M_i\rangle : M_i \text{ rejects } \langle M_i\rangle\}$
↳ explicitly by looping

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Clm. any TM M, $A \neq L(M)$

Pf. $M(\langle M\rangle)$ accept $\Rightarrow$ $\langle M\rangle \notin A = L(M) \ni \langle M\rangle$

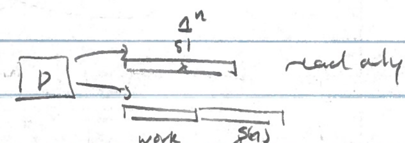
assume $A = L(M)$ for contradiction

$M(\langle M\rangle)$ reject $\Rightarrow$ $\langle M\rangle \in A = L(M) \not\ni \langle M\rangle$. ☒

idea: check if M_i acc/rej $\langle M_i\rangle$ while using $o(s(n))$ space
↳ only needs $O(s(n))$ space.

Pf: will give TM D, $A := L(D)$
- D runs in space $O(s(n))$ [complexity]
- $A \notin \mathrm{SPACE}(o(s(n)))$ [by construction] [correctness, requires analysis]

D = "on input x:
1) construct $1^{s(n)}$. $n = |x|$



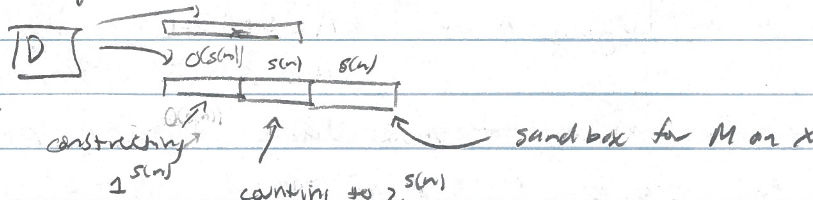
[uses space constructibility]

2) if $x \neq \langle M\rangle 10^k$ some M, k reject [allow asymptotic behavior]

3) simulate M on x ← - for $2^{s(n)}$ steps - using $s(n)$ tape cells → reject if bounds exceeded
M acc x $\Rightarrow$ accept
M rej x $\Rightarrow$ reject

Clm. D runs in space $s(n)$ [complexity]

Pf. always halts



[space constructibility]

[correctness: need to show simulation finishes]

Prop. exists universal TM U that can simulate any other
if M acc/rej x in time t, space s
$\Rightarrow$ U acc/rej $\langle M, x\rangle$ in time $O(c_M \cdot t \cdot s)$
space $O(c_M \cdot s)$
↳ "constant" depending on M.

[intuition: CPU (hardware) can run any software]

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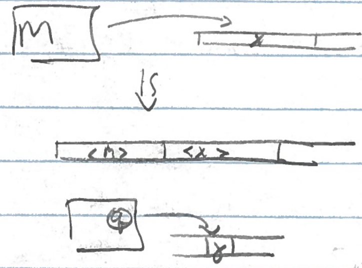
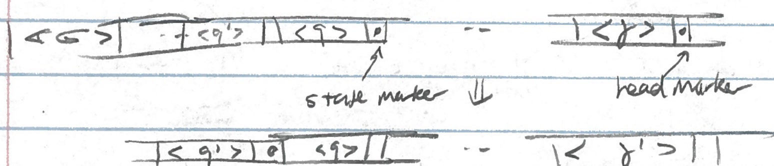
key points: U fixed TM, fixed tape alphabet Γ_U] also for state
tape alphabet Γ_M of M may be large] space

Sketch:

single step



$\delta(q,\gamma) \to (q',\gamma',L)$ ⇓



space: $O(c_M \cdot s)$

time: $\underbrace{O(c_M \cdot s)}_{\text{simulation per step}} \cdot \underbrace{t}$ ← # steps until acc/rej

$\boxtimes$

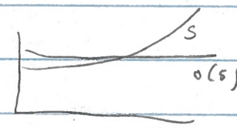
Clm: $A \notin SPACE(o(s(n)))$

Pf: M TM running in space $o(s(n))$

⇒ D simulating M on $\langle M\rangle 10^k$ takes $O(c_M \cdot o(s(|\langle M\rangle 10^k|)))$ space ← not "constant"

$= \underbrace{o(c_M \cdot s(|\langle M\rangle 10^k|))}$

⇒ $\exists k_0$ st all $k > k_0$



$< s(|\langle M\rangle 10^k|)$

and $2^{O(o(c_M \cdot s(n)))} < 2^{s(n)}$

⇒ at most $s(n)$ space used, $2^{s(n)}$ steps [time vs space bound]

⇒ D finishes simulating M on $\langle M\rangle 10^k$ any $k > k_0$

M acc x ⇒ D reject x

M rej x ⇒ D accept x

⇒ $A = L(D) \neq L(M)$. $\boxtimes$

Cor: $NL \subseteq SPACE(\log^2 n) \subsetneq SPACE(n) \subseteq PSPACE$ [can check space constructibility]

Thm [Gap Thm]: exist $s(n)$ st. $SPACE(s(n)) = SPACE(2^{s(n)})$ & space constructibility is necessary

= Questions?

thm (time hierarchy): $t(n): \mathbb{N} \to \mathbb{N}$ w/ ← in binary

$t(n)$ time constructible: $1^n \to \langle t(n)\rangle$ computable in $O(t(n))$ steps

exists language A st $A \in TIME(t(n))$

$A \notin TIME(o(\frac{t(n)}{\lg t(n)}))$

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Pf: as before, but now be careful w/ time

D = " on input x, $n = |x|$

1) compute $t(n)$, and reject if more than $\frac{t(n)}{\lg t(n)}$ steps in below
2) if $x \neq \langle M \rangle 10^k$ some M, k, reject
3) simulate M on x
M acc $x \Rightarrow$ reject
correctness: as before $\quad$ M rej $x \Rightarrow$ accept.

complexity: D runs in $t(n)$ time?

A: above universal simulation too slow
[but could use it to get weaker result]

$\langle M \rangle$ is "constant"

idea: - put $\langle M \rangle$ and $\frac{t(n)}{\lg t(n)}$-step clock on $O(\lg t(n))$ size tape

- move 2nd tape alongside first interleaved
← $O(\lg t(n))$ work per step

$\Rightarrow \frac{t(n)}{\lg t(n)}$ steps of $O(\lg t(n))$ work $\Rightarrow O(t(n))$ steps
[see book for more

Cor: $P \neq EXP = TIME(2^{poly(n)})$

today: - time, space diagonalization
next time: relativization