cs579: Computational Complexity Assigned: *Mon., Apr. 9, 2018*

Problem Set #6

Prof. Michael A. Forbes Due: *Mon., Apr. 23, 2018 (3:30pm)*

1. (Normal Form for Formulas) Given an unbounded fan-in $\{\text{AND}, \text{OR}, \text{NOT}\}$-formula of size-$s$, where size here is the number of $\{\text{AND}, \text{OR}\}$-gates, show that there is an equivalent formula of size $s' \le s$ where all negations occur at the bottom of the formula, and all $\{\text{AND}, \text{OR}\}$-gates have fan-in ≥ 2. Show that s' is bounded by the number of leaves of the resulting formula.

2. (Upper Bounds for Parity) Construct a $O(n^2)$-size $\{\text{AND}, \text{OR}, \text{NOT}\}$-formula for the parity function.

3. (Formulas vs Circuits)

   (a) Show that a fan-in-2 size-s $\{\text{AND}, \text{OR}, \text{NOT}\}$-circuit of depth-$d$ has a formula of size $2^{O(d)}$.

   (b) Show that an unbounded fan-in size-s $\{\text{AND}, \text{OR}, \text{NOT}\}$-circuit of depth-$d$ has a formula of size $s^{O(d)}$.

4. (Constant-Depth Upper Bounds for Parity) In this exercise, you will construct constant-depth $\{\text{AND}, \text{OR}, \text{NOT}\}$-formulas for the parity function. All size bounds will be measured in terms of the number of leaves.

   (a) Show that parity can be computed by a $O(n2^n)$-size CNF, and a $O(n2^n)$-size DNF.

   (b) By using divide-and-conquer, show that parity can be computed by a $2^{O(\sqrt{n})}$-size depth-4 formula where the output-gate is an AND-gate. Prove the same bound when the top gate is an OR-gate.

   (c) For every constant d, show that parity can be computed by a $2^{O(d \cdot n^{\frac{1}{d-1}})}$-size depth-$d$ formula.

Some hints.

2. Start with $n = 2$, then use divide and conquer.

4(c). Use (1) and push negations to the bottom. Use that $\text{AND}(\text{AND}(x, y), \text{AND}(z, w)) = \text{AND}(x, y, z, w)$.