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2018-01-17.4 →
2018-01-22.2 ←
2018-01-22.1
cs 579

cs 579: Computational Complexity: Lecture 2

admin: ps 1 out, due 02-05 3:30pm
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today: Turing Machines [review]
time
time hierarchy

what is computation?

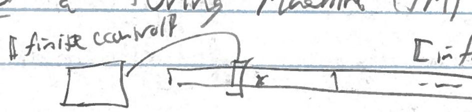
Q: can we schedule all courses, no conflicts?

A:
- decision [is there a solution] ← our focus
- search [find a solution] ← our focus
- verify [verify a solution] ← our focus
- optimization [best solution]
- counting: [how many solutions]

def: a language is $L \subseteq \{0,1\}^*$

Q: given $x \in \{0,1\}^*$, $x \in L$? [need model of computation]

def: a Turing Machine (TM) is



formally:
- Q set of states $= \{q_{start}, q_{acc}, q_{rej}, \ldots\}$
- Γ tape alphabet
- δ transition function
  δ: (current symbol, state) → (head movement, new symbol, new state)

a TM computes by:
- tape initialized to x
- head placed at start of tape
- iterate δ until reach q_{acc}/q_{rej}

language L of TM M, $L(M)$: M on x reaches q_{acc} ⟹ $x \in L$
q_{rej} ⟹ $x \notin L$
loops ⟹ $x \notin L$

more general: $q_{output} \in Q$, M on x reaches q_{output} ⟹ output = tape contents

def: a TM M runs in time $t(n): \mathbb{N} \to \mathbb{N}$ if for all $x \in \{0,1\}^n$,
M on x halts in $\le t(n)$ steps [worst case notion]
space $s(n)$ " uses $\le s(n)$ tape cells

def: $\mathrm{TIME}(t(n)) = \{$ lang L : L decided by TM in time $\in O(t(n))\}$ [big Oh]
$\mathrm{SPACE}(s(n)) =$ space $\in O(s(n))\}$
L, $s(n) \ge n$ [will relax next week]

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Prop: TIME$(f(n)) \subseteq$ SPACE$(f(n))$ [one cell per time cell]

philosophy

Church Turing **Thesis**: anything "computable" is computable by a TM
assumption, backed by evidence

Extended " : efficient, efficiently [true "for now"]
↳ possibly inconsistent w/ quantum

def: $P = \bigcup_k \text{TIME}(n^k)$ [efficient in theory]

Q: what is in P? [no one really knows..]

def: $\text{EXP} = \bigcup_k \text{TIME}(2^{n^k})$

Prop: $P \subseteq \text{EXP}$

Q: = ?

thm: ≠ [today]

= Questions

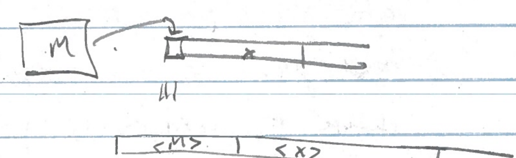
simulation: hardware ≡ software (non fact)
TM M — description ⟨M⟩ of TM M ← some encoding in binary

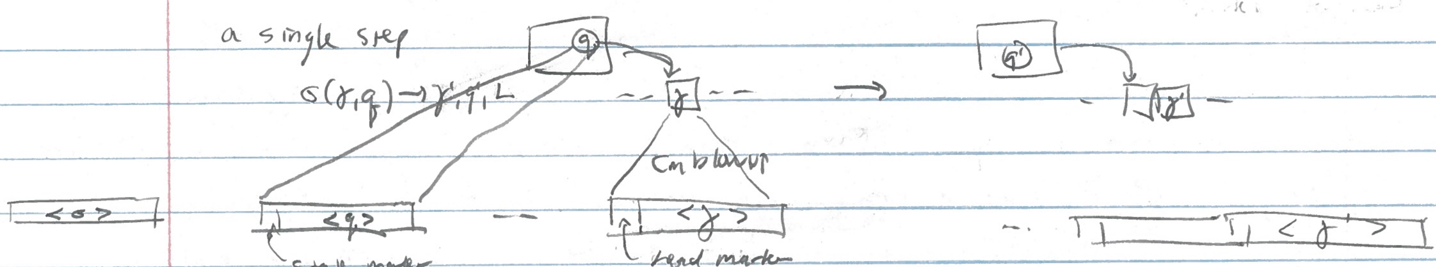
thm: exists universal TM U that can simulate any other
ie M acc/rej x in time t space s
⇒ U acc/rej ⟨M,x⟩ in time $O(c_M \cdot t \cdot s) \le O(c_M \cdot t^2)$
space $O(c_M \cdot s)$
↳ constant depending on M

intuition: CPU can run any programming language

key points: U is fixed, fixed tape alphabet Γ_U] also for steps space
tape alphabet Γ_M of M may be large

Pf sketch:





time: $O(c_M \cdot s) \cdot t$

space $O(c_M \cdot s)$ ∎

Rmk: w/ two tapes can get $O(c_M \cdot t \cdot \lg t)$ time [not qualitatively crucial]

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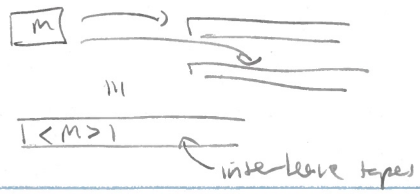
Rmk: same result if M has more tapes



time hierarchy theorem

Thm: for any "nice" functions $f(n), g(n)$ w/ $f(n)^2 = o(g(n))$ [can do better]

$TIME(f(n)) \subsetneq TIME(g(n)^2)$ [more time more power]

Thm [Gap theorem]: "not nice" function w/ $TIME(f(n)) = TIME(2^{f(n)})$

def: $f(n)$ is time constructible if map $1^n \mapsto \langle f(n) \rangle$ ← binary is

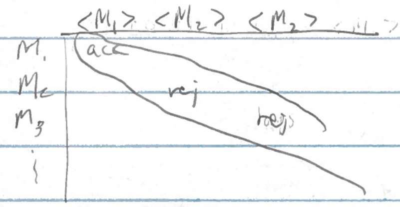
- computable in time $O(f(n))$ [use output of TM]
- monotonically increasing

Pf: [time travel is impossible]

recall: given TM M, deciding if M halts on x is undecidable

want: ... efficiently ... efficiently

idea:



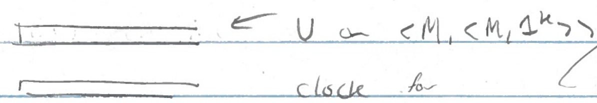
D rej acc acc ... ⟹ $L(D) \neq L(M_i)$ all i

D = " on input $\langle M, 1^k \rangle$: [reject if not of this form]

1) use U to simulate M on $\langle M, 1^k \rangle$ for $g(|\langle M, 1^k\rangle|)$ steps of real time
2) accept $\langle M, 1^k \rangle$ iff ↑ rejects

Claim: $L(D) \in TIME(g(n)^2)$

Pf: poset two tapes



⟹ $O(g(n)^2)$ steps
↑ hidden constant only depends on U

Rmk: need g time constructible

Clm: $L(D) \notin TIME(f(n))$

Pf: suppose not $L(D) \in TIME(f(n))$

$f(n)$ steps of M on $\underbrace{\langle M, 1^k \rangle}_{\sim n}$ takes $O(c_M f(n)^2)$ steps of time on U

$f^2 = o(g)$ ⟹ k large ⟹ $f(|\langle M, 1^k\rangle|)^2 \ll g(|\langle M, 1^k \rangle|)$

⟹ $< g(|\langle M, 1^k\rangle|)$

⟹ U simulates M on $\langle M, 1^k \rangle$ and does opposite!

Rmk: the "constant" c_M is not constant.

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Q: what is $L(D)$? [unnatural]

def: BOUNDED HALTING$_f$ = $\{\langle M, x\rangle$: M accepts $\langle M, x\rangle$ in $\le f(|x|)$ steps$\}$

cor: f time constructible ⟶ $\in$ TIME(f^2) [universal simulation + clock]
$\notin$ $\sqrt{f}$ [diagonalization]

Thm [SPACE hierarchy]: $f(n), g(n)$ space constructible $f = o(g)$ ⇒

SPACE($f(n)$) $\subsetneq$ SPACE($g(n)$)

(space simulation has no penalty)

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next time: nondeterminism
NP
reductions
NP-completeness
NP-intermediate