CS 579. Computational Complexity
Problem Set 5

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due April 14, 2017

Collaboration Policy: The homework can be worked on in groups of up to 3 students each (2 would be optimal, but 1 and 3 are both accepted).

One submission per team is sufficient. Please write the solution for each of the problems on a separate sheet of paper. Write your team names and netids on each submission and please **staple** all the sheets together.

Submissions should be written in LaTeX, unless your handwriting is indistinguishable from LaTeX.

Homework is due before the end of class, April 14. Only one late homework per person will be allowed. If you submit more than one homework late, you will get no grade for the excess late homeworks.

Problem 1 (40 pts.)

A directed graph is d-regular if every vertex has in-degree d and out-degree d. Prove that the st-Connectivity problem in directed regular graphs is in **L**. (Hint: give a reduction to the undirected case.)

Problem 2 (20 pts.)

Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix with real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and orthonormal eigenvectors $x_1, \ldots, x_n$. Let $\text{span}(v_1, \ldots, v_k) = \left\{ \sum_{i=1}^{k} \alpha_i v_i \;\middle|\; \alpha_i \in \mathbb{R} \right\}$ for any set of vectors $v_1, \ldots, v_k$ for any $k \in \mathbb{N}$.

Prove that

$$\begin{aligned}
\lambda_{k+1} &= \min_{x \perp \mathrm{span}(x_1, x_2, \ldots, x_k)} \frac{x^T A x}{x^T x} \\
&= \min_{x \in \mathrm{span}(x_{k+1}, x_{k+2}, \ldots, x_n)} \frac{x^T A x}{x^T x} \\
&= \max_{x \perp \mathrm{span}(x_{k+2}, x_{k+2}, \ldots, x_n)} \frac{x^T A x}{x^T x} \\
&= \max_{x \in \mathrm{span}(x_1, x_2, \ldots, x_{k+1})} \frac{x^T A x}{x^T x}.
\end{aligned}$$

Problem 3 (20 pts.)

One problem with the above characterization of eigenvalues of the previous problem is that it requires us to know the eigenvectors $x_1, \ldots, x_{k-1}$ (or eigenvectors $x_{k+1}, \ldots, x_n$ in order to compute the eigenvalue λ_k. The Courant-Fischer theorem gives us a more general way of computing these eigenvalues. Let A be a symmetric real matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and corresponding orthonormal eigenbasis $x_1, \ldots, x_n$. Prove that the following is true:

$$\begin{aligned}
\lambda_{k+1} &= \min_{\substack{W \subseteq \mathbb{R}^n: \\ \dim(W) = k+1}} \max_{x \in W} \frac{x^T A x}{x^T x} \\
&= \max_{\substack{W \subseteq \mathbb{R}^n: \\ \dim(W) = n-k}} \min_{x \in W} \frac{x^T A x}{x^T x}
\end{aligned}$$

Problem 4 (20 pts.)

Suppose that G is connected and Δ is the maximum degree of any vertex in G. Let λ_1 be the maximum eigenvalue of the adjacency matrix A of G. Prove that $\lambda_1 = \Delta$ if and only if G is Δ-regular.