# Computational Complexity

Lecture 0

# Computation

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- Theory of computation/computational complexity is to computer science what theoretical physics is to electronics

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  - How much "resource" is sufficient/necessary

# Computational Complexity

of Problems

in Models of computation

w.r.t Complexity measures

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    - A Boolean function evaluation (TRUE/FALSE)

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- Non-uniform computation: circuit families

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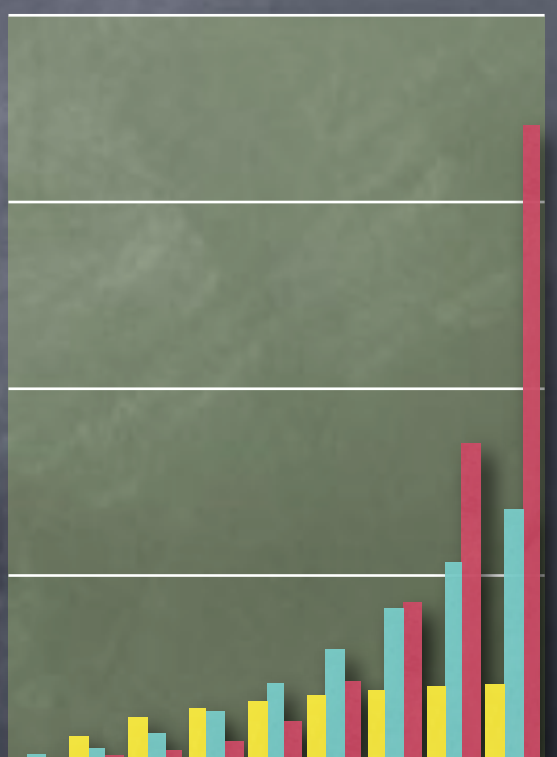

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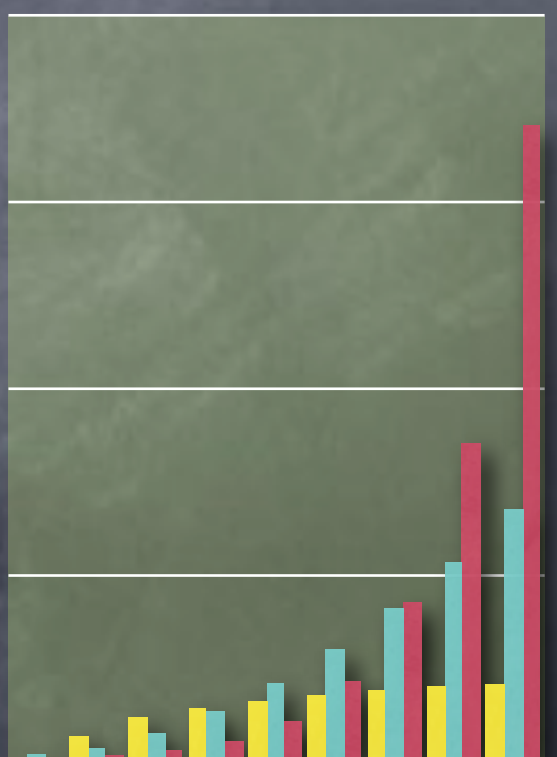

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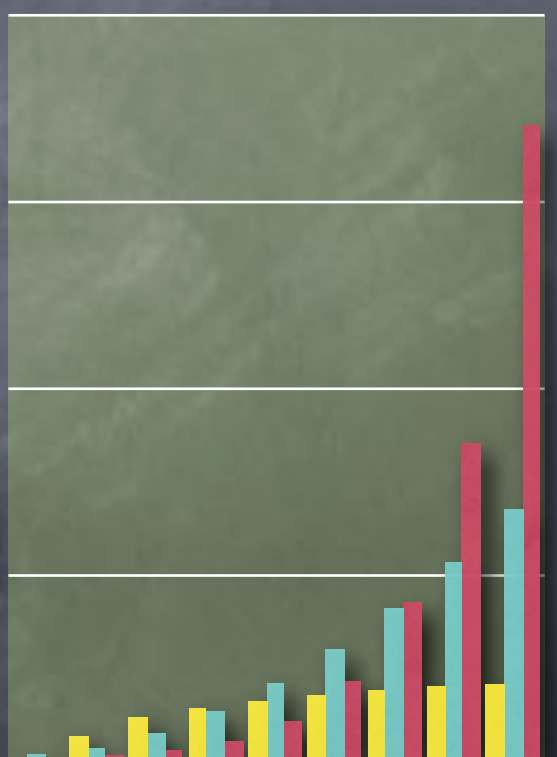

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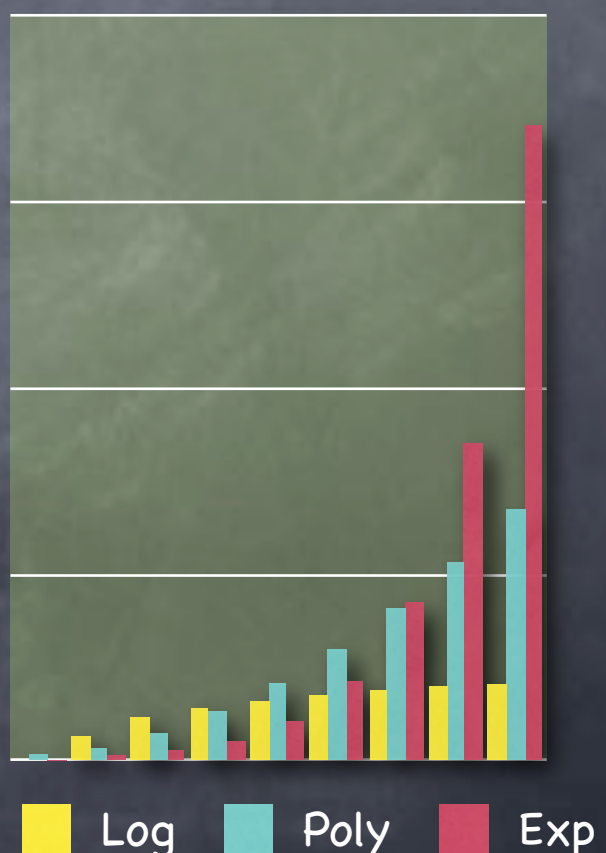

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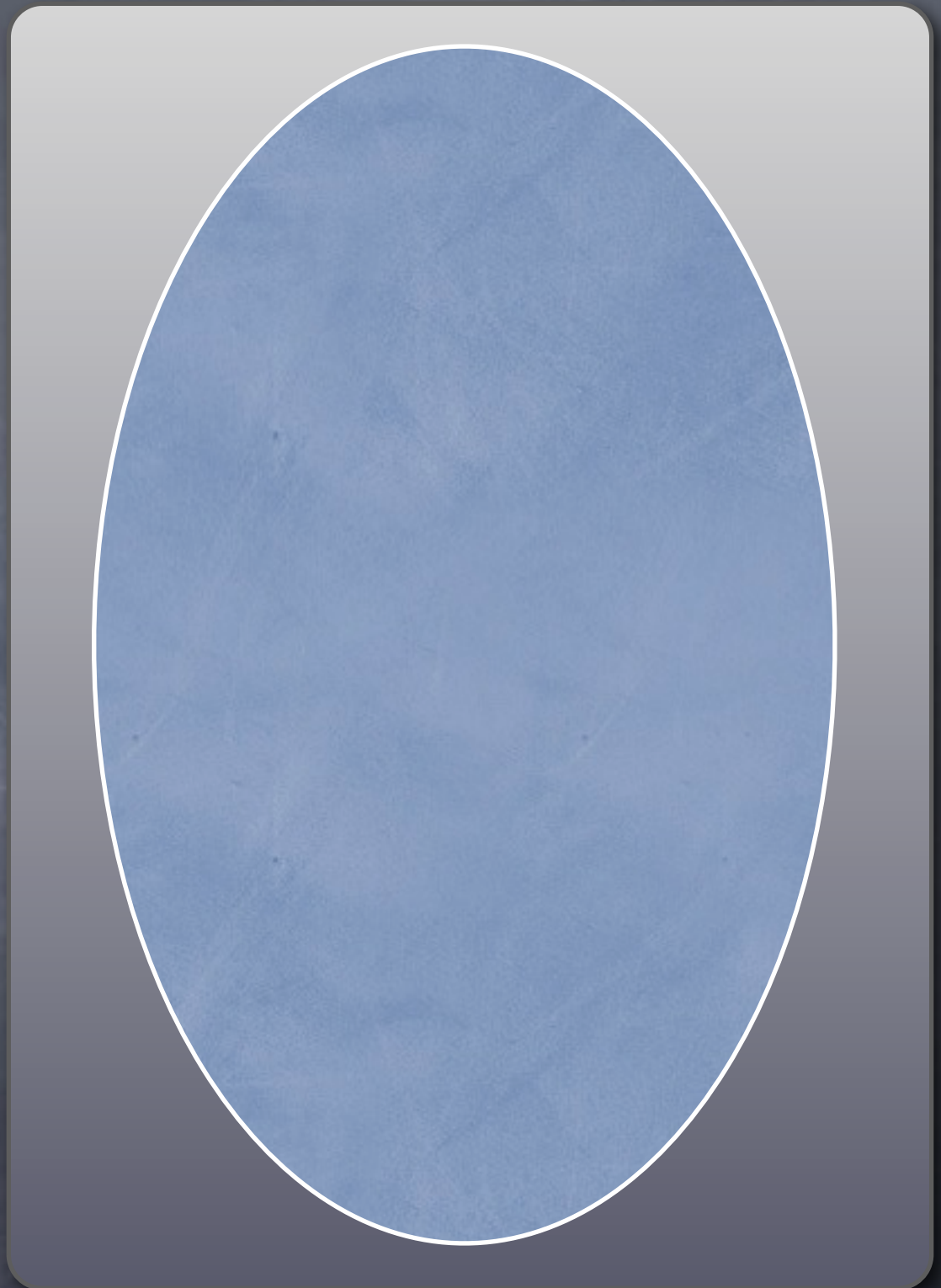

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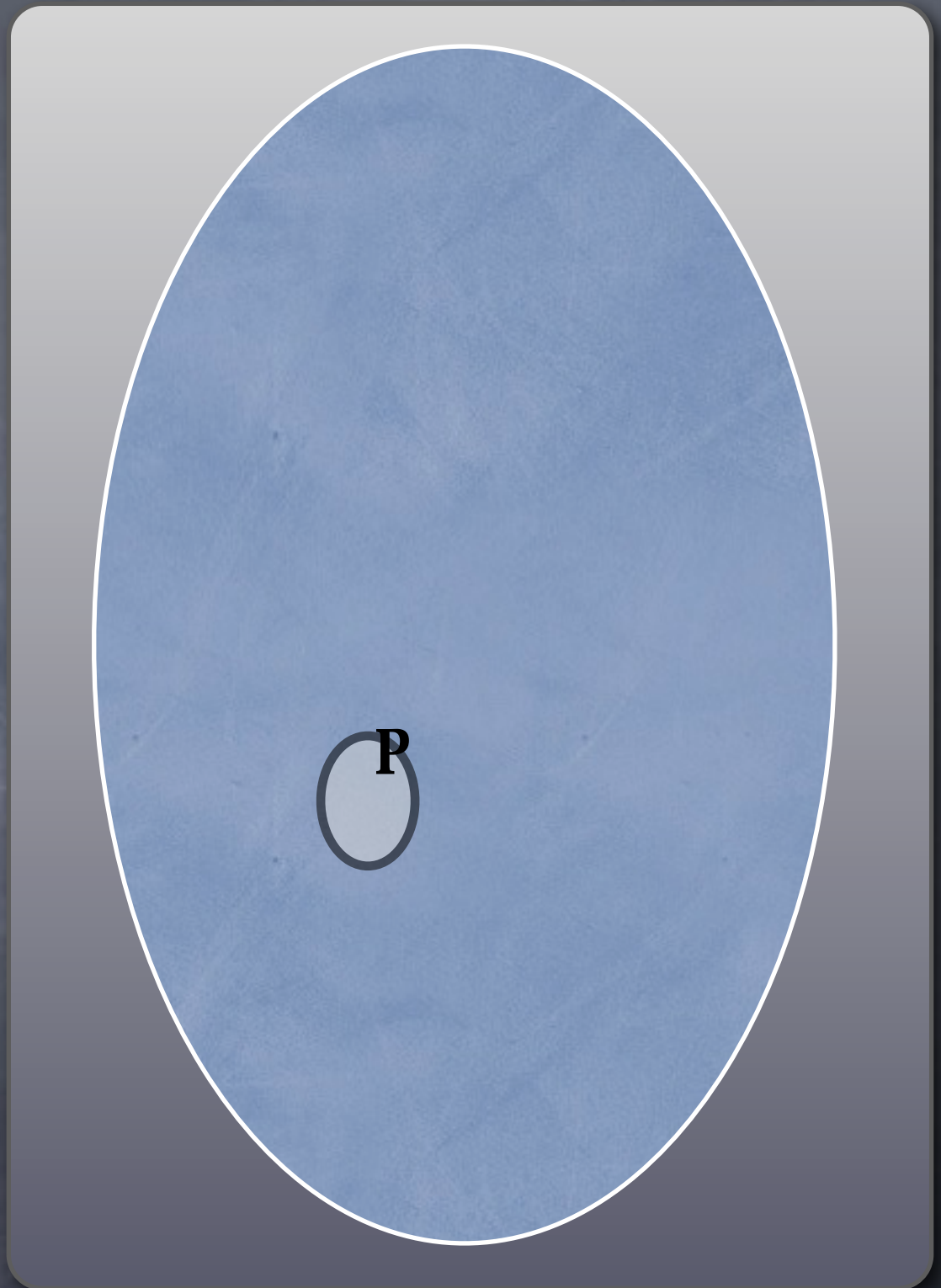

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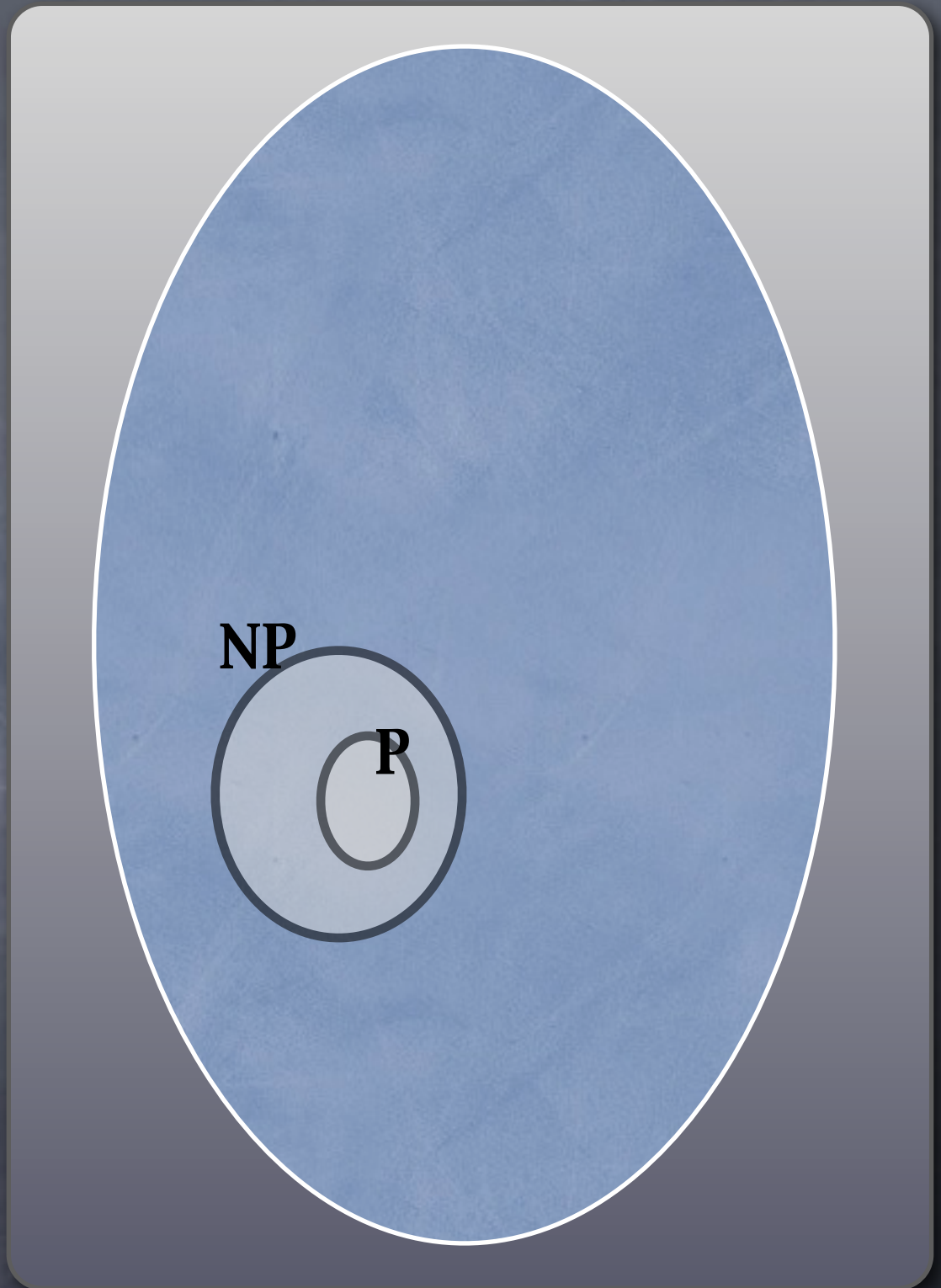

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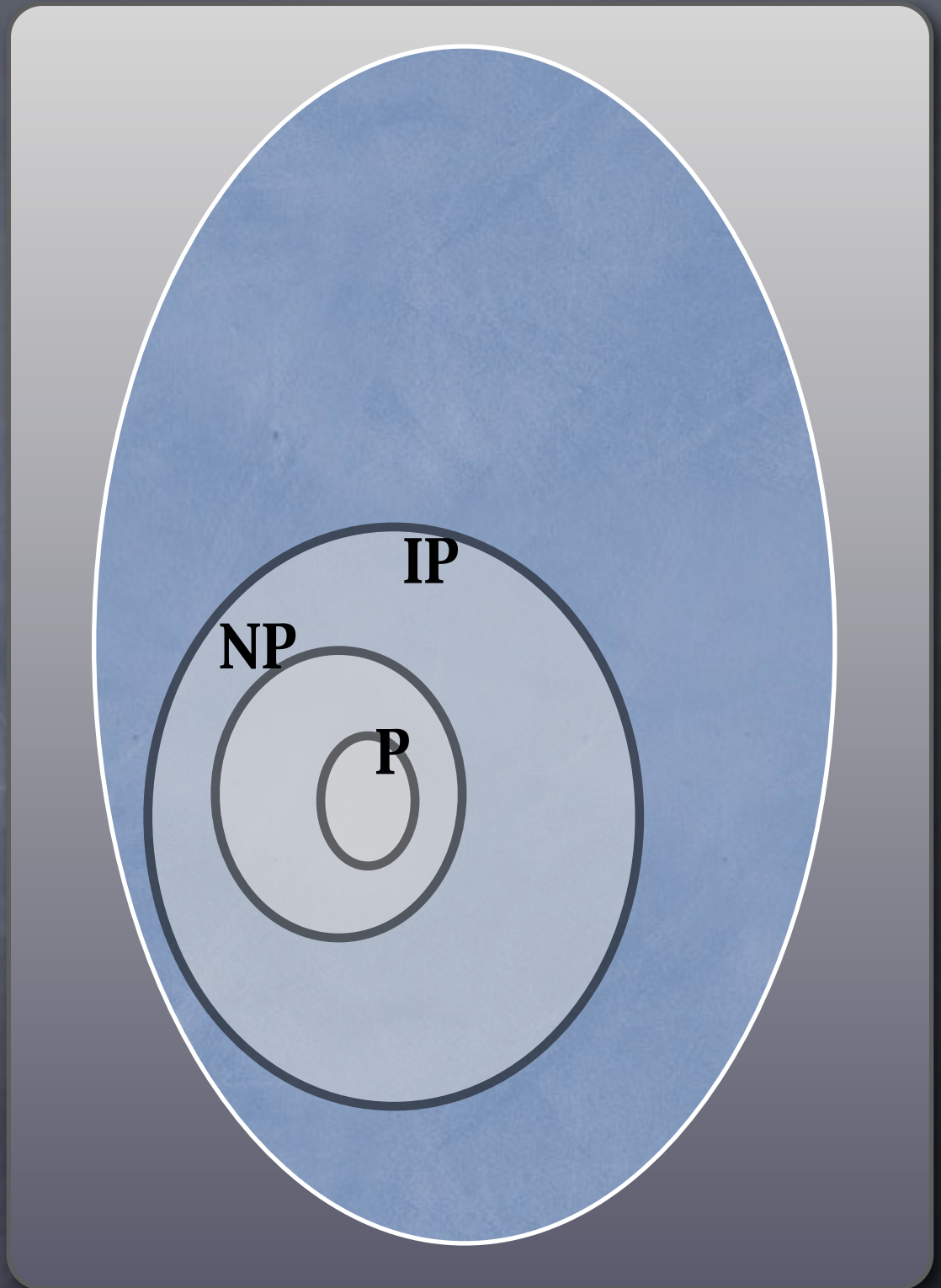

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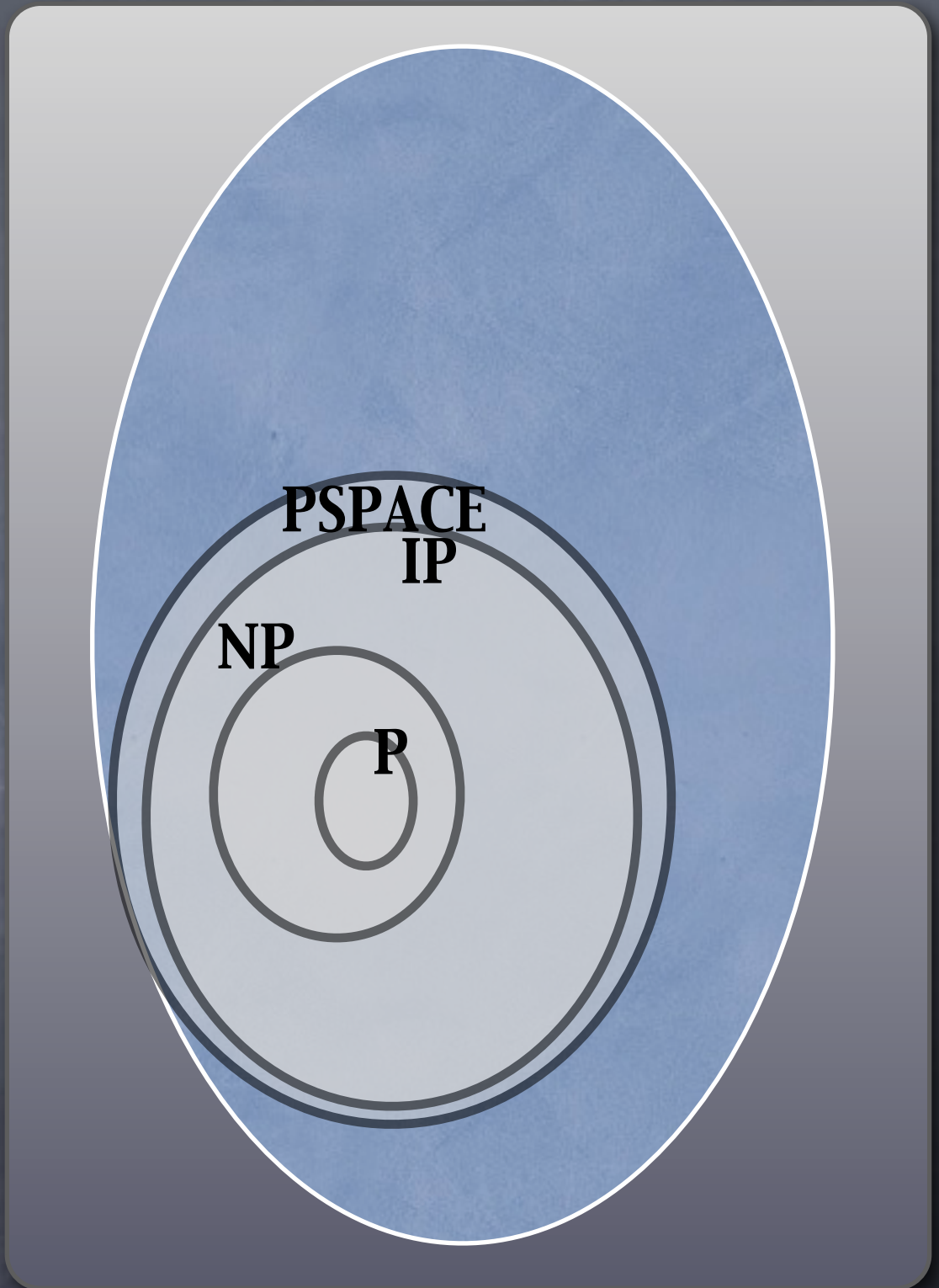

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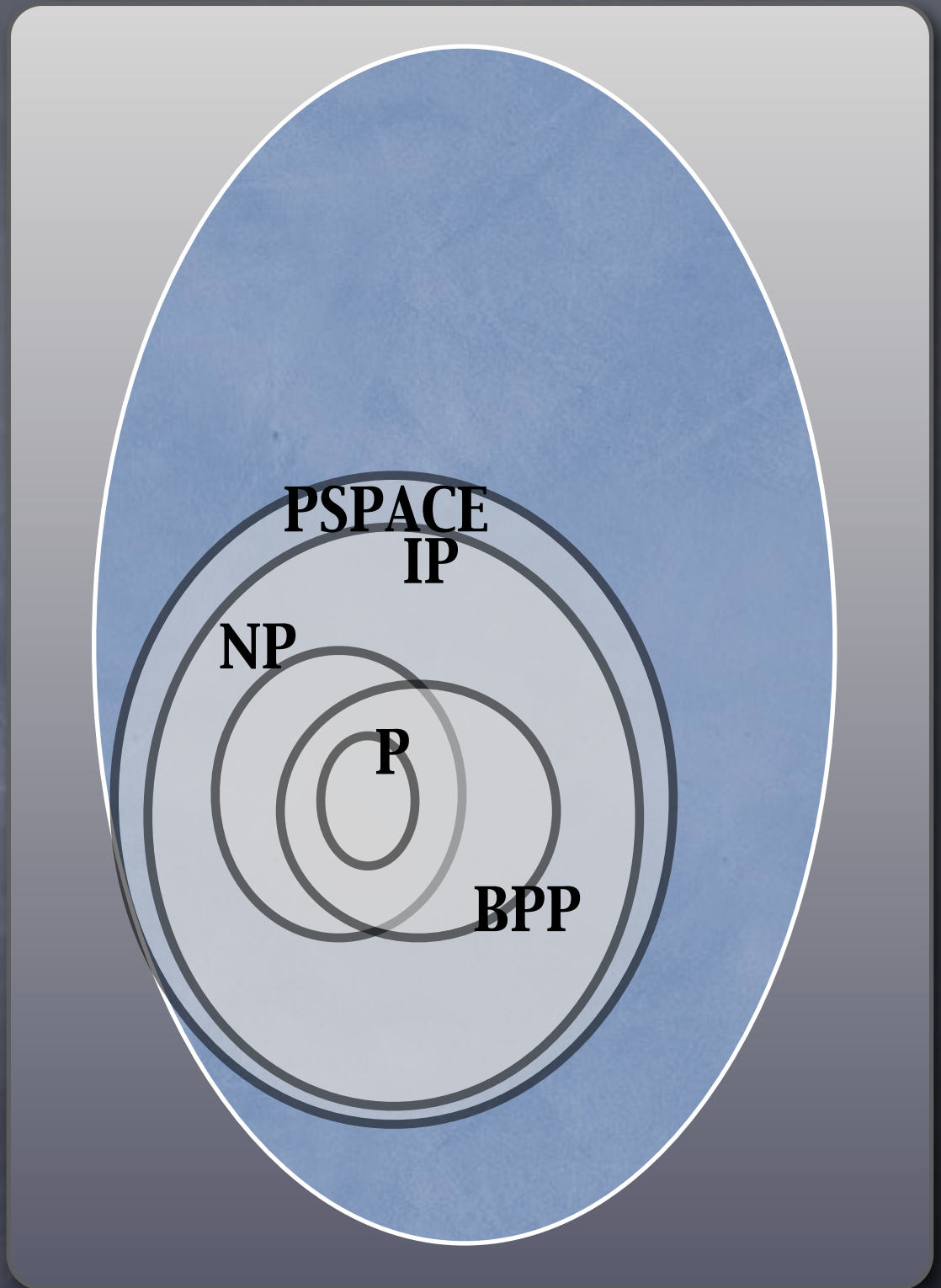

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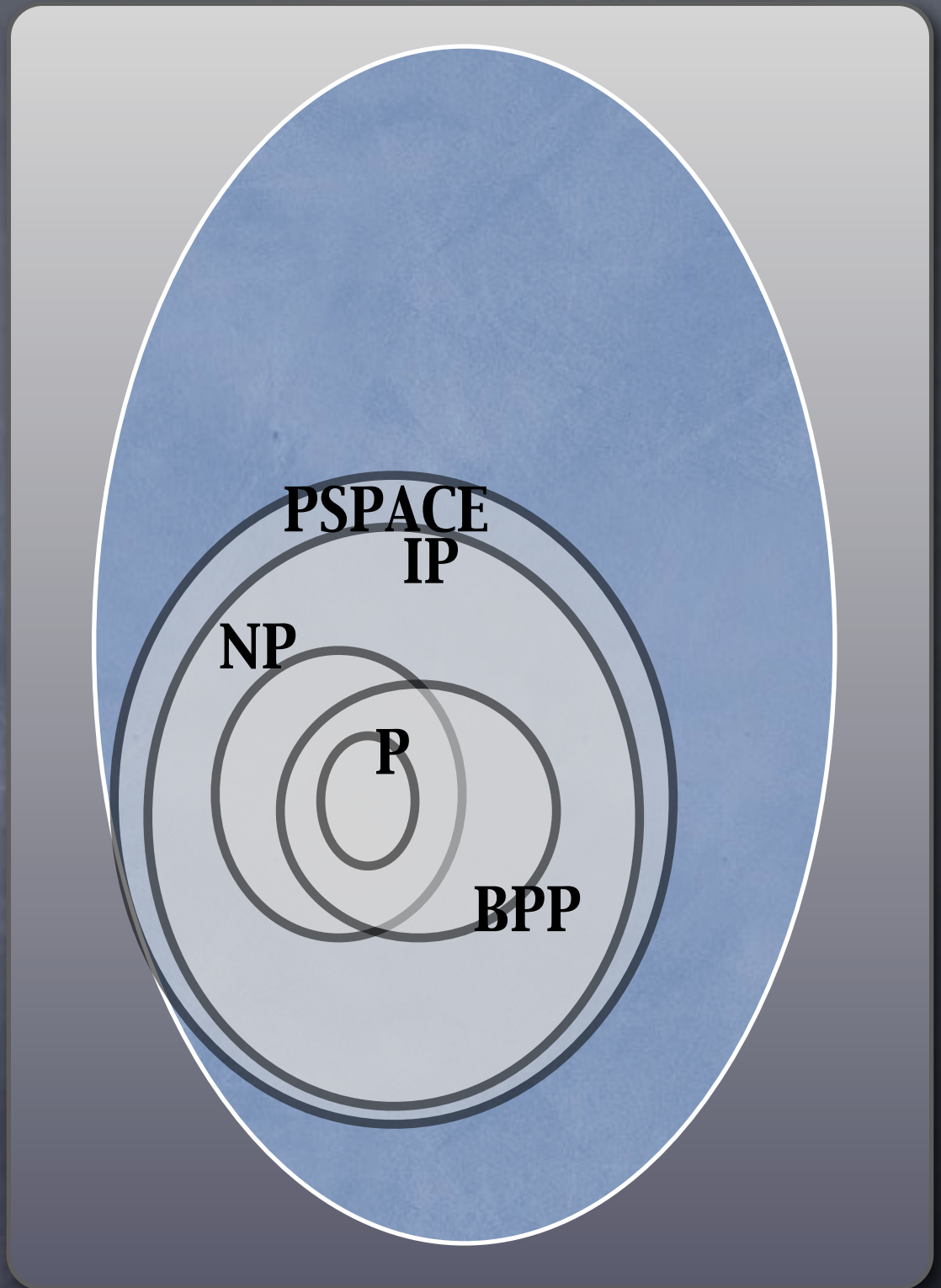

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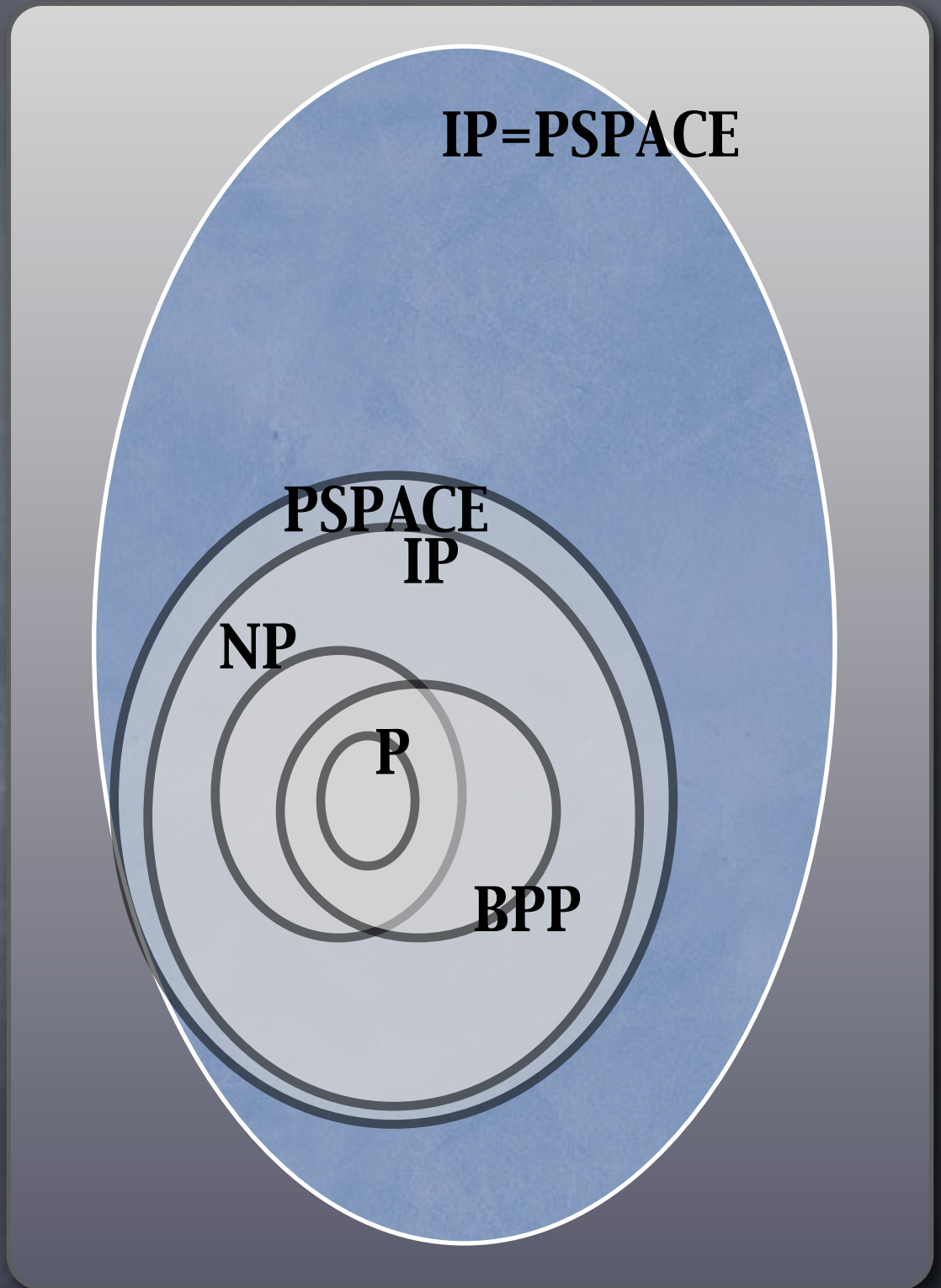

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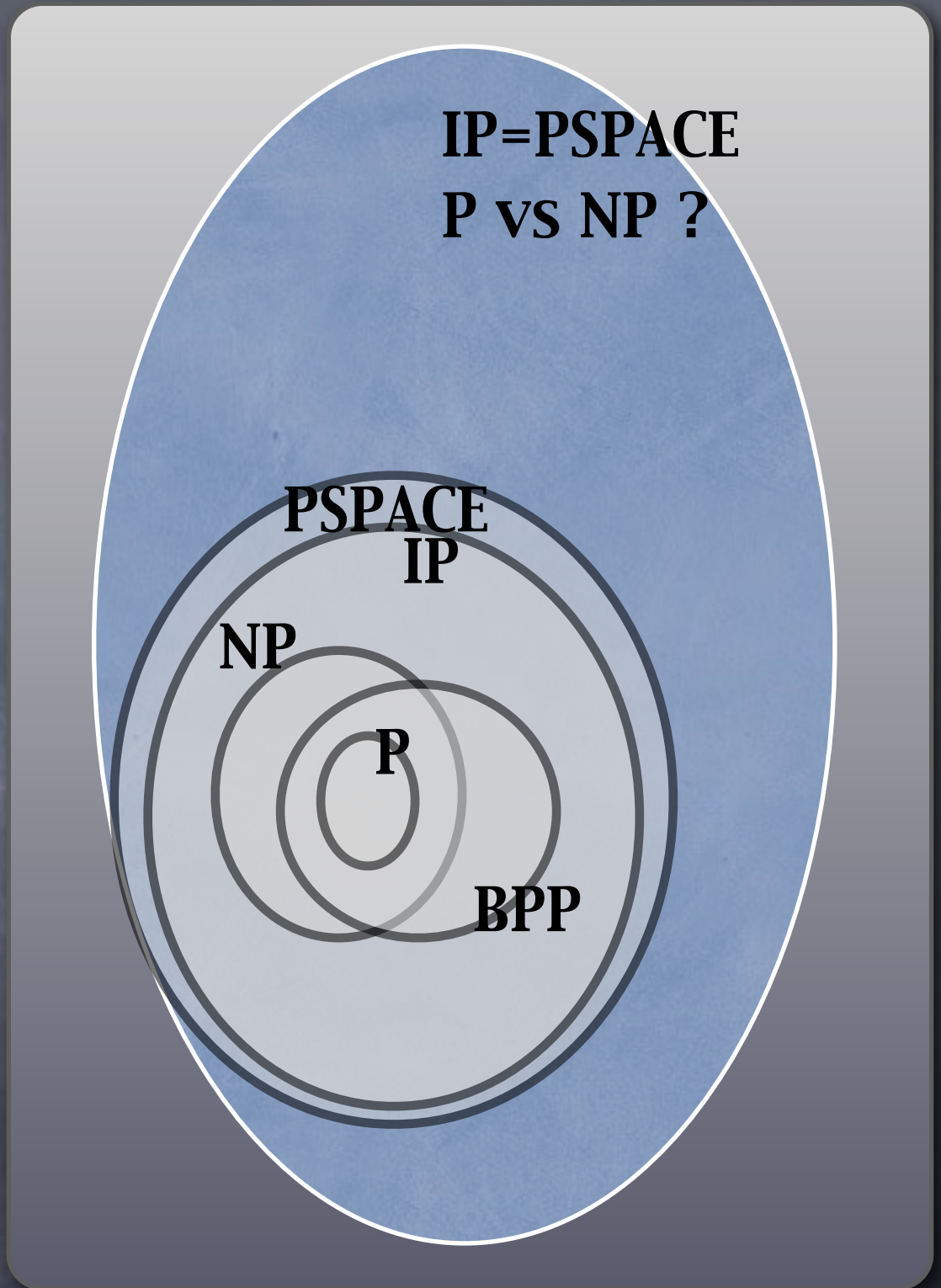

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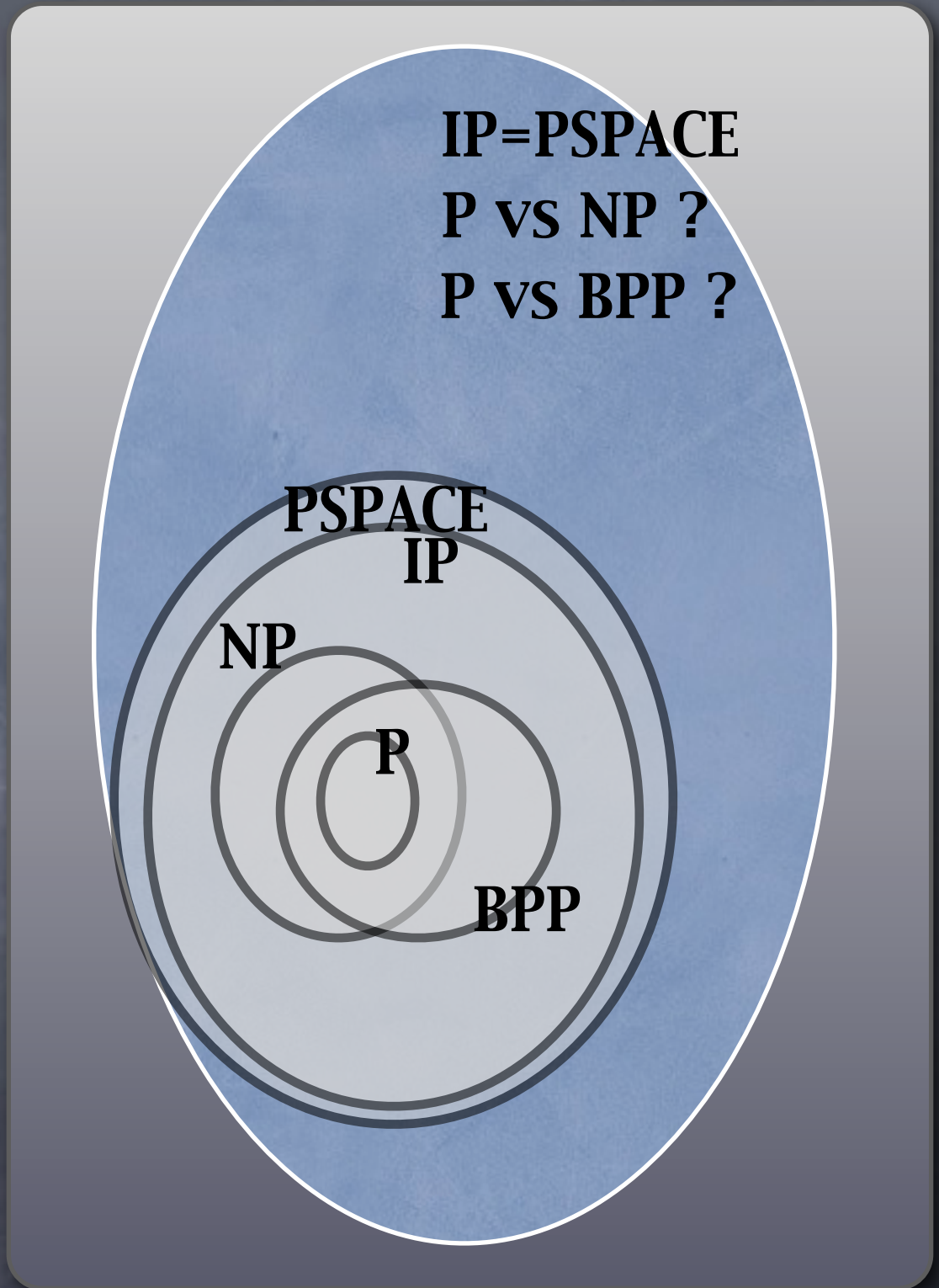

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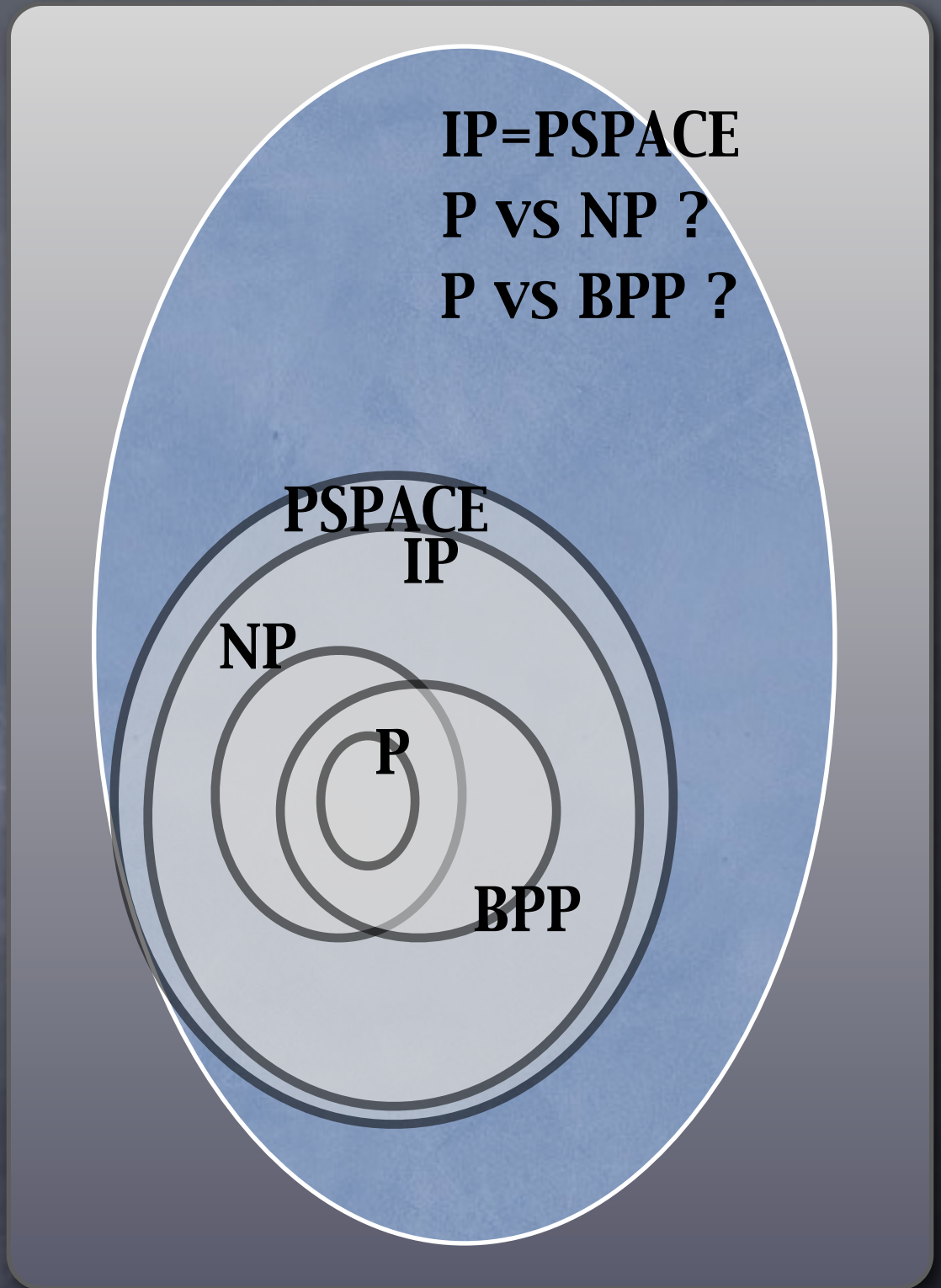

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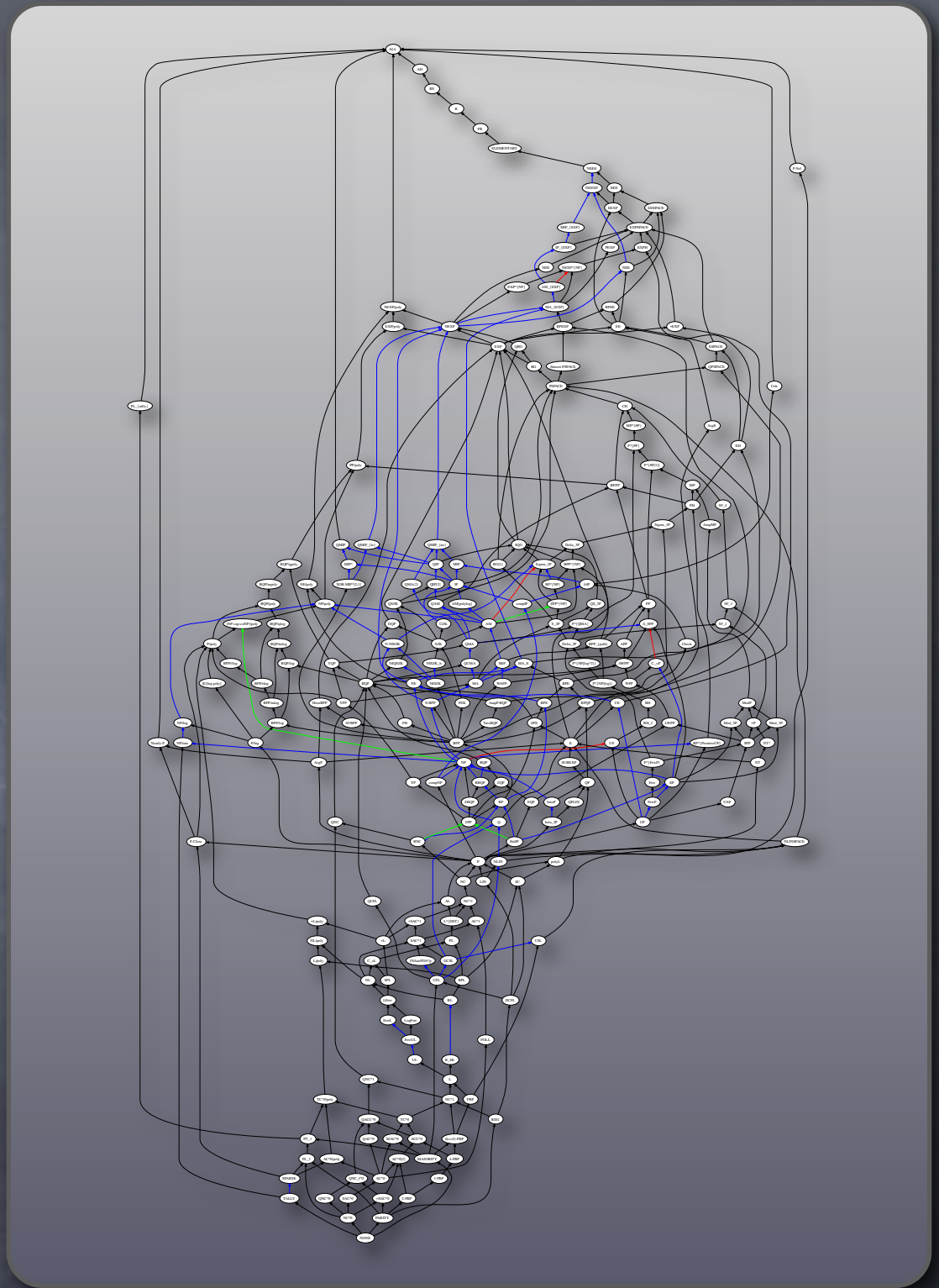

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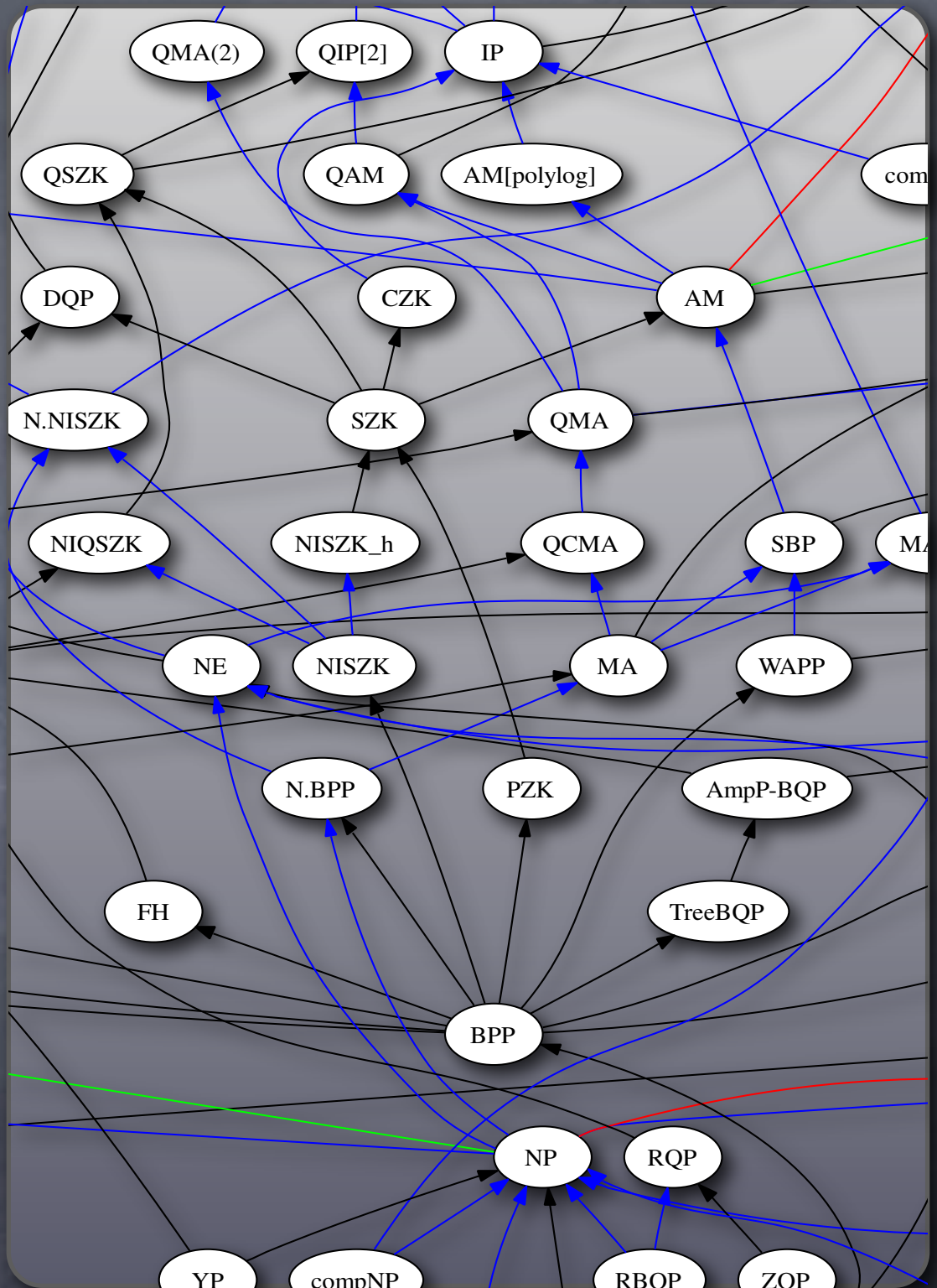

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- Various connections: time vs. space, randomness vs. hardness

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- Contrast with Information Theory: limited cryptography if no computational constraints on adversary

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- Recent results...

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