Complexity Homework 1

Released: January 25, 2011

Due: February 8, 2011

For problems that involve nondeterministic complexity classes, the solutions maybe simpler when phrased in terms of "certificates" (instead of non-determinism).

Problem 1:

(a) Let L_1, L_2 be languages in **NP**. Are $L_1 \cup L_2$ and $L_1 \cap L_2$ necessarily in **NP**?

(b) Let L_1, L_2 be languages in **NP**. Show that $L_1 L_2$ and L_1^* are in **NP**.

(c) Let L_1, L_2 be languages in **P**. Show that $L_1 L_2$ and L_1^* are in **P**.

(d) Let L_1, L_2 be languages in $\mathbf{NP} \cap \mathbf{co\text{-}NP}$. Show that their symmetric difference

$$L_1 \oplus L_2 \stackrel{\text{def}}{=} \{x \mid x \text{ is in exactly one of } L_1, L_2\}$$

is also in $\mathbf{NP} \cap \mathbf{co\text{-}NP}$.

Problem 2:

(a) Show that the halting problem is **NP**-hard. Is it **NP**-complete?

(The halting problem is given by the language $H = \{(\langle M \rangle, x) \mid M \text{ is a TM that halts on input } x\}$. You may recall that H is *undecidable*.)

(b) Show that $\overline{\mathsf{SAT}}$ (the complement of SAT) is **NP**-hard *under Cook reductions*. That is, every language in **NP** reduces to $\overline{\mathsf{SAT}}$ via a Cook reduction. (On the other hand, we believe $\overline{\mathsf{SAT}}$ is not NP-hard (under Karp reductions). If it were, then $\mathbf{NP} = \mathbf{co\text{-}NP}$.)

Problem 3:

Show that the following two statements are equivalent (we don't know if they are true):

(a) Every unary[1] language in **NP** is also in **P**.

(b) $\mathbf{DTIME}(2^{O(n)}) = \mathbf{NTIME}(2^{O(n)})$ (these classes are called **E** and **NE**, respectively).

Hint: It takes $\Theta(\log n)$ bits to encode the number "n" in binary.

Problem 4:

Give a parsimonious Karp reduction from SAT to $\mathsf{3SAT}$.

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[1] A language is *unary* if it is a subset of $\{1\}^*$ — that is, it only uses one symbol of the alphabet.

Problem 5:

In this problem, we analyze a reduction from $\textsf{3SAT}$ to the following language:

$$\textsf{MAX-2SAT} = \{(\phi, k) \mid \phi \text{ is a 2-CNF formula, and there is an assignment that satisfies at least } k \text{ clauses}\}$$

Our reduction is the following: Given a $\textsf{3SAT}$ instance ϕ, we will output a $\textsf{MAX-2SAT}$ instance (ϕ', k), where ϕ' is a 2-CNF formula. To construct ϕ', do the following: for each clause $(x \vee y \vee z)$ in ϕ, add the following 10 clauses to ϕ' (where w is a fresh variable for each clause):

$$(x), (y), (z), (\neg x \vee \neg y), (\neg y \vee \neg z), (\neg x \vee \neg z), (w), (x \vee, \neg w), (y \vee \neg w), (z \vee \neg w)$$

Find a value of k such that $(\phi', k) \in \textsf{MAX-2SAT}$ if and only if $\phi \in \textsf{3SAT}$. Prove the correctness of the reduction.

Problem 6 (Extra credit):

Show that $\textsf{2SAT}$ is in $\mathbf{P}$.

Hint: Consider a directed graph with all the literals as nodes, and edges as implications ($(x \vee y)$ corresponds to $(\neg x \Rightarrow y)$ and $(\neg y \Rightarrow x)$). Look to derive contradictions of the form $(\neg x \Rightarrow x)$ and $(x \Rightarrow \neg x)$. What do such contradictions tell you about a possible satisfying assignment?

Problem 7 (Extra credit) [See Arora-Barak (web-draft) Chapter 2, Exercise #13]:

Show that if there is a unary language that is $\mathbf{NP}$-complete, then $\mathbf{P} = \mathbf{NP}$.

Problem 8 (Extra credit):

Consider the following language:

$$\textsf{MAX-CUT} = \{(G, k) \mid G \text{ is a multigraph with a cut of size at least } k\}$$

A *cut* in a graph is a partition of its vertices into two parts. The size of the cut is the number of edges which "cross" the cut (whose endpoints are in opposite parts). A multigraph means we allow duplicate edges.

We now analyze a reduction from $\textsf{MAX-2SAT}$ to $\textsf{MAX-CUT}$. Given an instance (ϕ, k) of $\textsf{MAX-2SAT}$, let n be the number of variables occuring in ϕ, and m the number of clauses. Consider the following graph:

G_ϕ is a graph with a vertices labeled x_i and $\neg x_i$ for each variable x occuring in ϕ, and two special vertices labeled T and F. We add $5m$ edges between T and F, and $5m$ edges between each pair $(x_i, \neg x_i)$ — see Figure 1. Then, for each clause $(x \vee y) \in \phi$, where x and y are literals, we add the following 7 edges (see Figure 2):

- $(x, y), (T, x), (T, y)$.
- Two copies of the edges (x, F) and (y, F).

(a) Show that in the largest cut in G_ϕ, T and F must be in opposite parts.

(b) Show that in the largest cut in G_ϕ, the vertices corresponding to x and $\neg x$ must be in opposite parts.

(c) Argue that $(\phi, k) \in \textsf{MAX-2SAT}$ if and only if $(G_\phi, 5m + 5mn + 4k + 2(m - k)) \in \textsf{MAX-CUT}$.

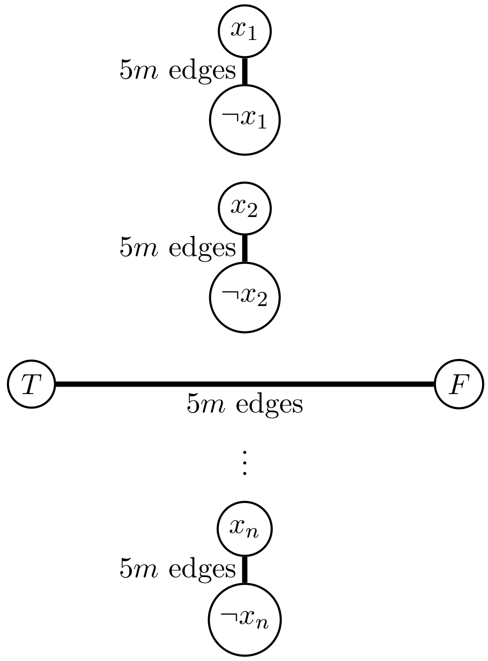

Figure 1: Starting graph for G_ϕ, where ϕ has n variables, $x_1, \ldots, x_n$.

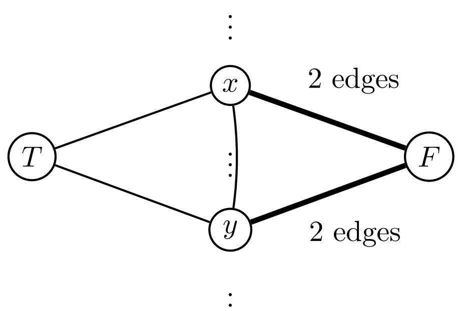

Figure 2: Edges to add for a clause of the form $(x \vee y)$