Complexity of Counting

Lecture 21

#P
FP

Turing Machines computing a (not necessarily Boolean) function of the input
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Writes the output on an output tape
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  - Writes the output on an output tape
- **FP**: class of efficiently computable functions
  - Computed by a TM running in polynomial time
Counting Problems
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Counting: Functions of the form \("\text{number of witnesses}\)\)
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e.g: Number of subgraphs of a given graph with some property (trees, cycles, spanning trees, cycle covers, etc.)
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e.g.: Number of satisfying assignments to a boolean formula

e.g.: Number of inputs in a language L that are less than x (lexicographically)
Class of functions of the form number of witnesses for an NP language
#P

Class of functions of the form **number of witnesses for an NP language**

\[ \#R(x) = |\{w: R(x,w)=1\}|, \text{ where } R \text{ is a polynomial time relation} \]
#P

Class of functions of the form \textbf{number of witnesses for an NP language}

\( \#R(x) = \left| \{ w : R(x,w) = 1 \} \right| \), where \( R \) is a polynomial time relation

\textbf{e.g.:} \( \#\text{SPANTREE}(G) = \) number of spanning trees in a graph \( G \)
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- e.g.: #SPANTREE(G) = number of spanning trees in a graph G
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- e.g.: #SAT(φ) = number of satisfying assignments of φ
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e.g.: \( \#\text{CYCLE}(G) \) = number of simple cycles in a directed graph \( G \)
e.g.: \( \#\text{SAT}(\phi) \) = number of satisfying assignments of \( \phi \)

Easy to see: \( \text{FP} \subseteq \#\text{P} \) (interpreting numbers as strings suitably)

[Exercise]
#P vs. NP
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- If \#P = FP, then P = NP
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- \( \#P \) “harder” than \( NP \)
  - If \( \#P = FP \), then \( P = NP \)
  - How much harder?
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  - If \( \text{#CYCLE} \in \text{FP} \), then \( P=NP \)
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Reduce HAMILTONICITY to #CYCLE: Given G, to construct G' such that #CYCLE(G') is “large” iff G has a Hamiltonian cycle.
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Reduce HAMILTONICITY to #CYCLE: Given G, to construct G' such that #CYCLE(G') is “large” iff G has a Hamiltonian cycle.

Replace each edge in G by a gadget such that each cycle in G becomes “many” cycles in G'.
#CYCLE $\in$ FP $\Rightarrow$ P=NP

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A single n-long cycle in G will result in more cycles in G′ than produced by all shorter cycles in G put together
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HAMILTONICITY(G) $\iff$ #CYCLES(G) $\geq n^{n^2}$
#P vs. PP
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Recall PP: x in L if for at least half the strings w (of some length) we have R(x,w)=1
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i.e., checking the most significant bits of $\#R$
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Recall: We already saw $NP \subseteq PP$
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  - PP as powerful as #P (and vice versa)
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#P ⊆ FP^{PP} [exercise] (and PP ⊆ P^{#P} [why?])
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#P ⊆ FP^{PP} [exercise] (and PP ⊆ P^{#P} [why?])

So if PP = P, then #P = FP (and vice versa)
#P completeness
f ∈ #P is #P-complete if any g ∈ #P can be Cook-reduced to f
\[ f \in \#P \text{ is } \#P\text{-complete if any } g \in \#P \text{ can be Cook-reduced to } f \]
#P completeness

- $f \in \#P$ is \#P-complete if any $g \in \#P$ can be Cook-reduced to $f$

- From parsimonious reduction of $g$'s NP problem to an NP-complete problem (w.r.t Karp-reductions)

Allows multiple oracle calls. Alternately, allow only one call.
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Other #P-complete problems whose decision problems are in P

Permanent (for binary matrices) is #P-complete
Permanent
Permanent

Permanent of a square matrix A
Permanent

- Permanent of a square matrix $A$
- If $A$ is binary (0,1 entries): $\text{perm}(A) = \text{number of perfect matchings in a bipartite graph } B_A$ whose adjacency matrix is $A$
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- Algebraically: $\text{perm}(A) = \sum_\sigma \prod_i A_{i,\sigma(i)}$ where $\sigma$ are permutations
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Permutations are cycle covers of complete directed graph
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Weight of a cycle cover $\sigma$, $W(\sigma) = \prod_i A_{i,\sigma(i)}$
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- Permutations are cycle covers of complete directed graph

- Weight of a cycle cover $\sigma$, $W(\sigma) = \prod_i A_{i,\sigma(i)}$

- $\text{Perm}(A) = \sum_{\sigma} W(\sigma)$ over all cycle covers $\sigma$ of directed graph $G_A$ (with edge-weights from $A$)
Permanent is \#P-complete
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- First will reduce \#SAT to permanent of an integer (not binary) matrix
Permanent is \#P-complete

- First will reduce \#SAT to permanent of an integer (not binary) matrix

- Plan: Given a SAT instance \( \varphi \) with \( m \) clauses, build an integer-weighted directed graph \( A_\varphi \) such that
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Almost Karp-reduction (need to rescale)
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Permanent is \#P-complete

*For each variable add a “variable gadget” and for each clause a “clause gadget”*

*Variable:* two possible cycle covers of weight 1 -- *uses* either all the true-edges or the false-edge

*Clause:* any cycle cover has to leave at least one variable-edge *free*

<table>
<thead>
<tr>
<th>Gadget:</th>
<th>Symbolic description:</th>
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<tbody>
<tr>
<td>variable gadget:</td>
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<tr>
<td>False edge</td>
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[Figures from the textbook]
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- XOR gadget (with negative edge weights):
Permanent is \#P-complete

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\begin{array}{c}
\text{u} \\
\text{v}
\end{array}
\rightarrow
\begin{array}{c}
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\]
Permanent is \#P-complete

- **XOR gadget** (with negative edge weights):
  - Replacing a pair of edges by an XOR gadget changes total weight of cycle covers using neither or both the edges to 0, and scales total weight of cycle covers using exactly one of them by 4.
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- **Final graph**
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  - Each satisfying assignment gives a cycle cover of weight $4^{3m}$
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Permanent is $\#P$-complete

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- Also, let $M$ be a power of 2 ($M = 2^k$). Replace $M$ by $\log M$ edges of weight 2 in series, each further replaced by +1 weight edges
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- #P complete problems
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- Next: Toda’s Theorem: PH ⊆ P^{#P} = P^{PP}