PCP

Lecture 26
And Hardness of Approximation

Promise Problems

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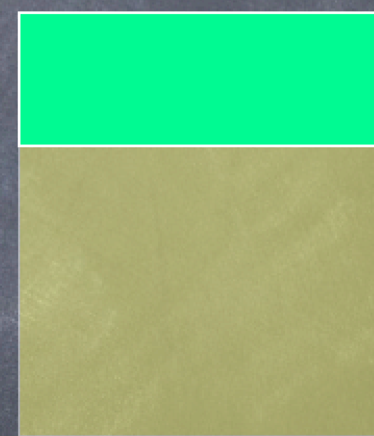

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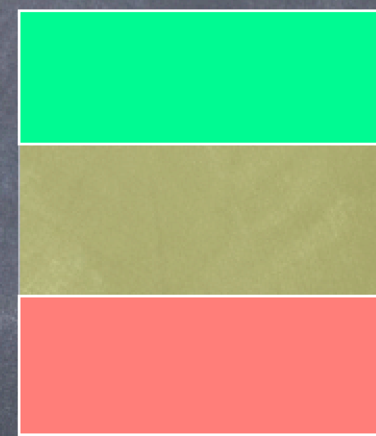

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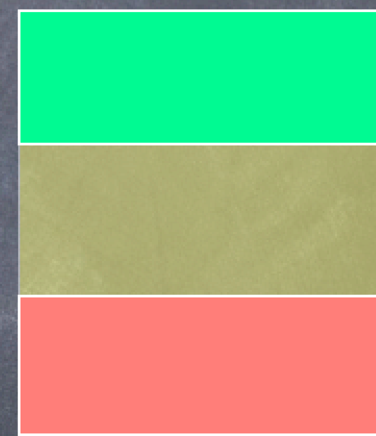

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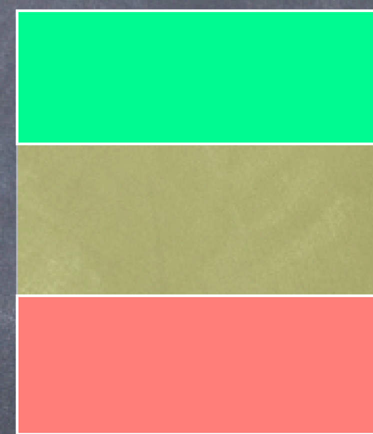

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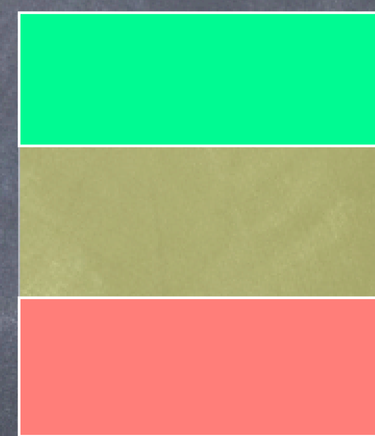
  - For inputs outside the two, don't care
    - We're "promised" that such inputs are not given



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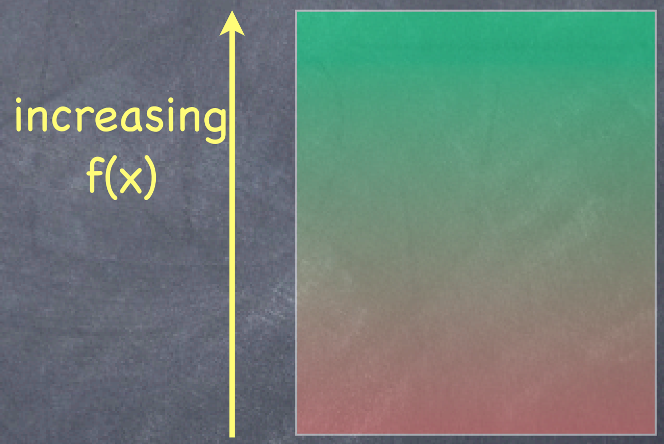

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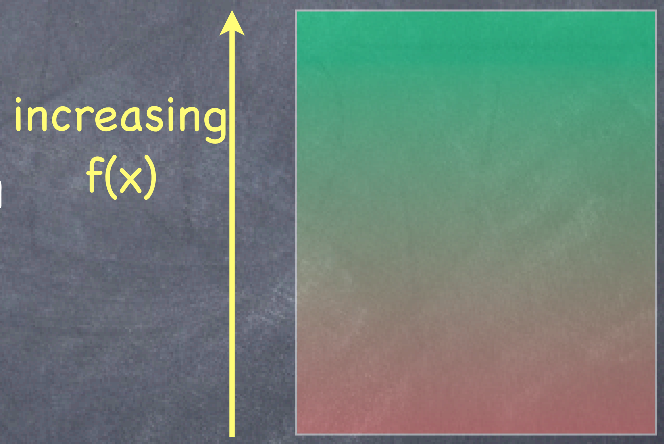

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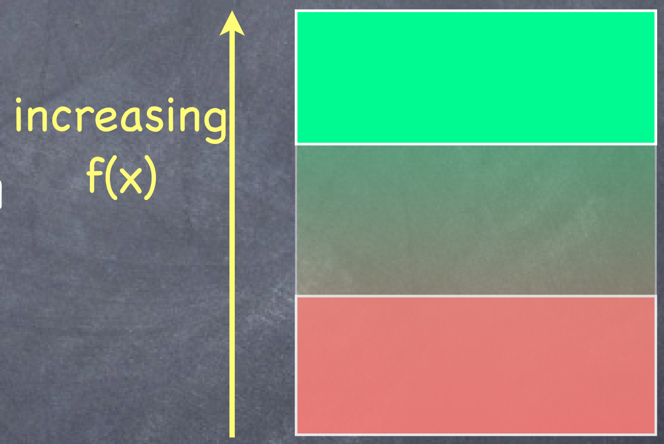

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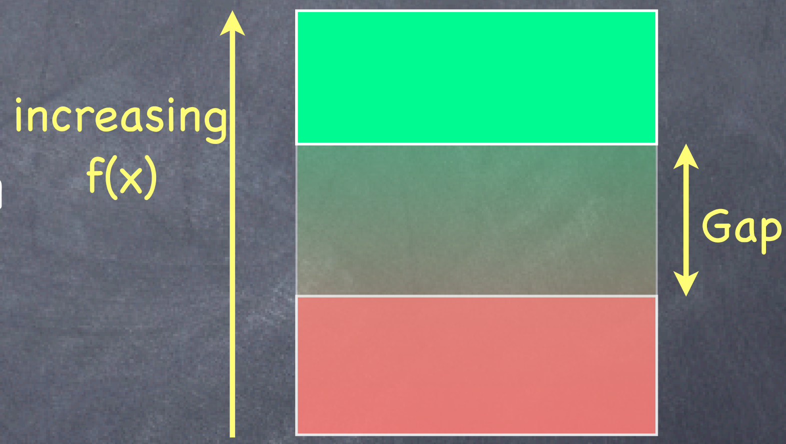

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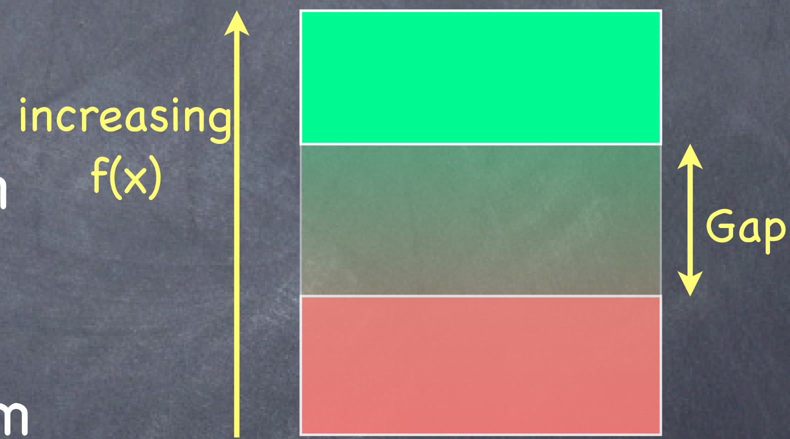

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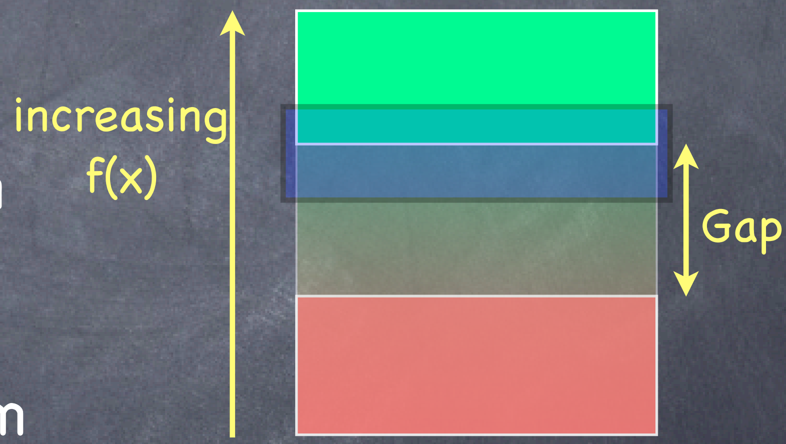

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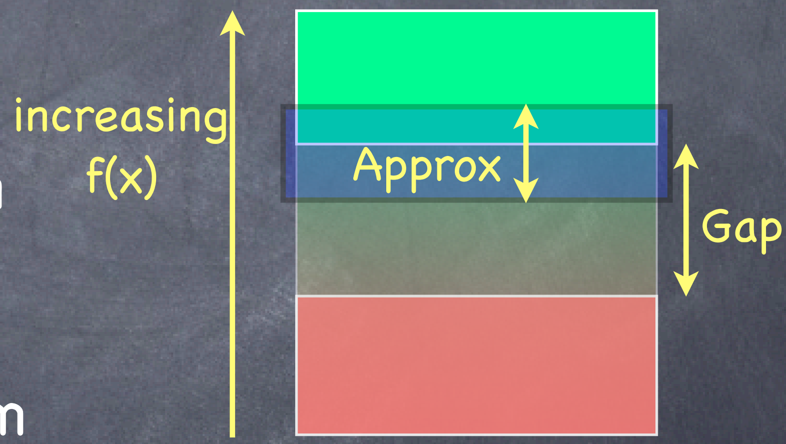

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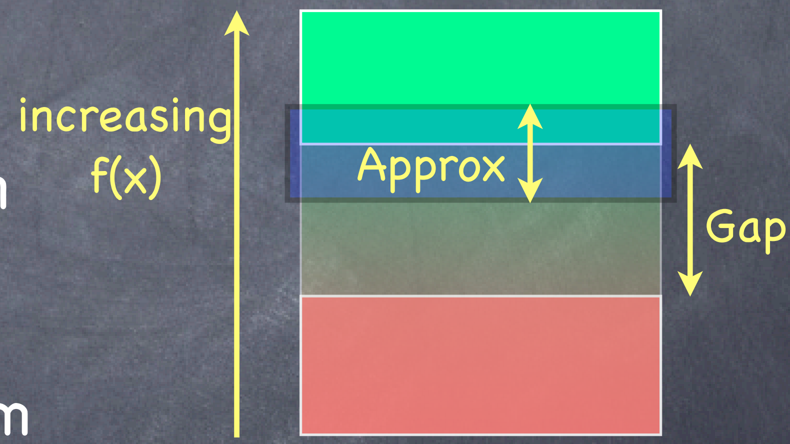

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  - PCP(log m,q): q-query (non-adaptive) verifier, tosses at most log m coins

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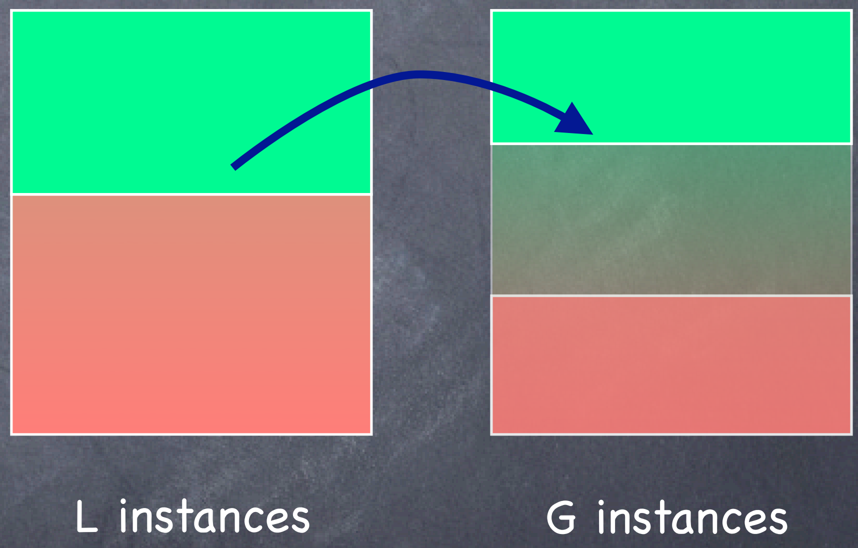

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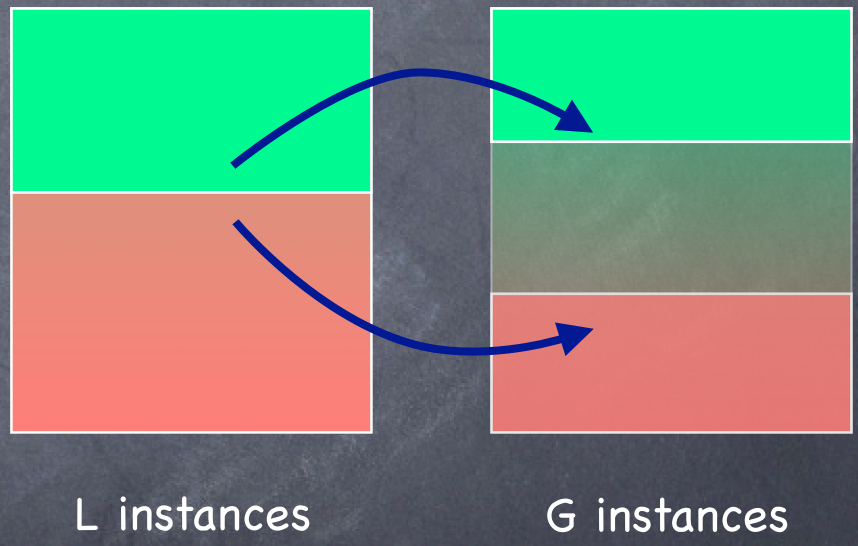

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  - Such that approximation for them imply approximation for Max-qCSPSat

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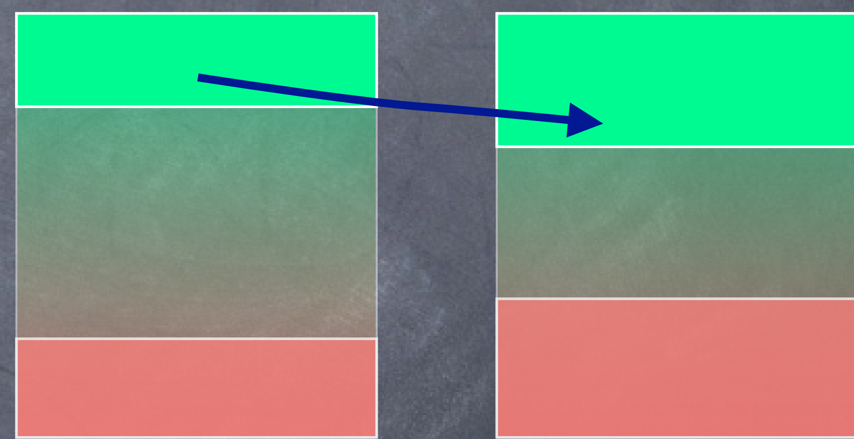

G_1 instances G_2 instances

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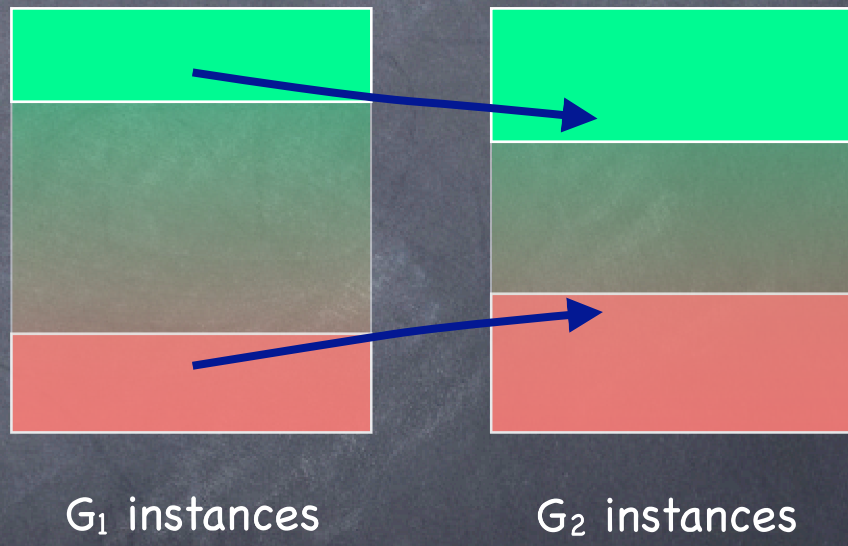

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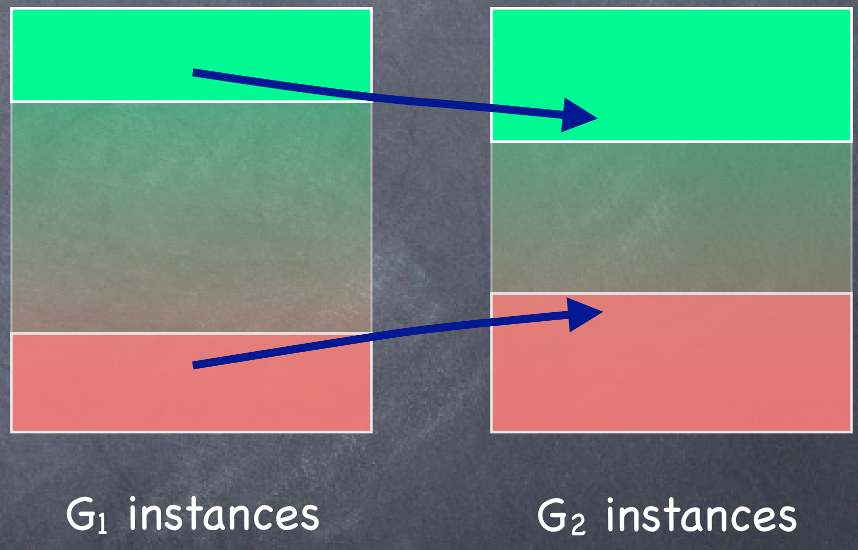

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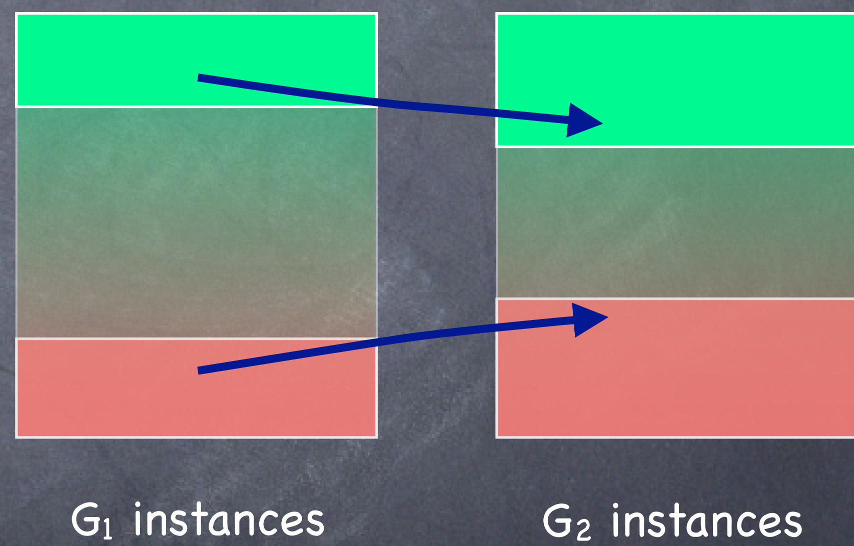

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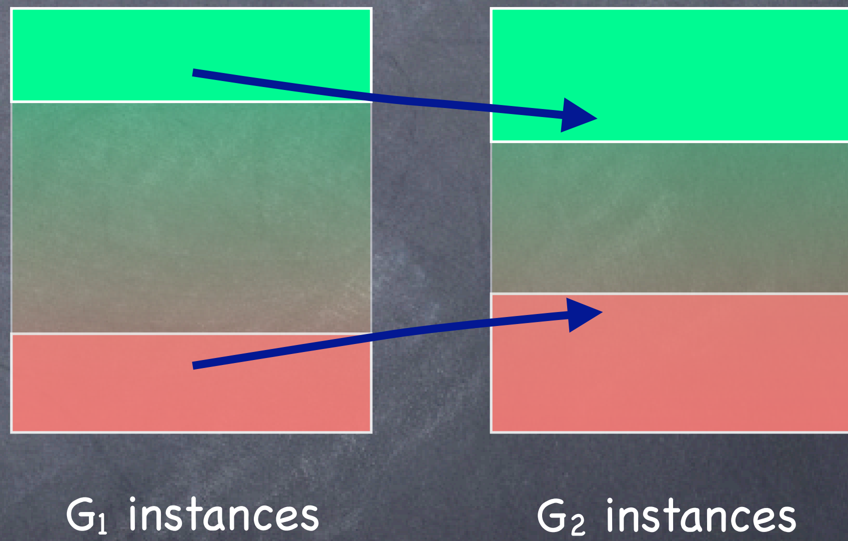
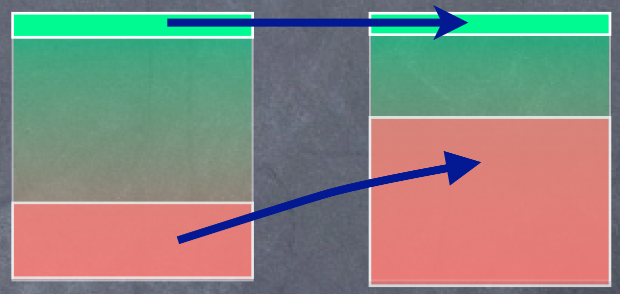
    - The bigger the gap in G_2 the larger the approximation factor shown hard



Max-qCSP to Max-3SAT



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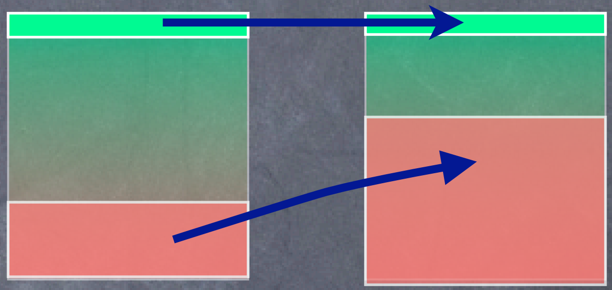

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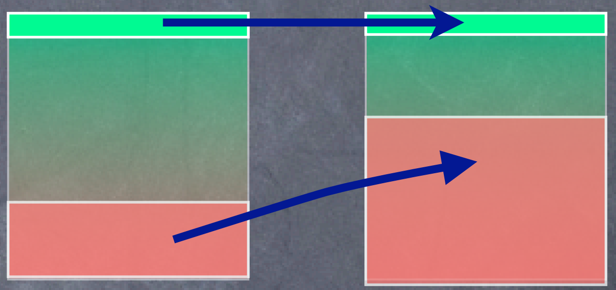

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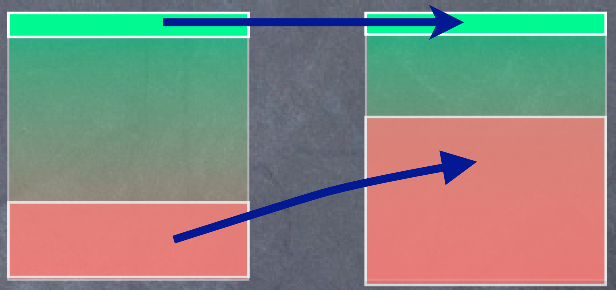

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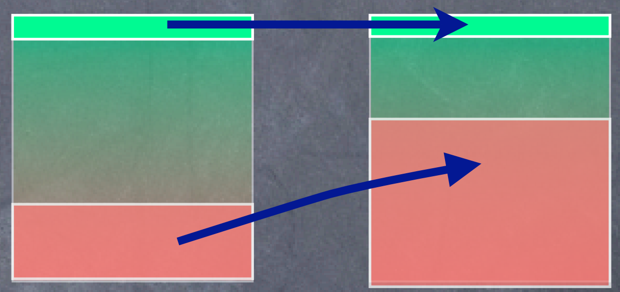

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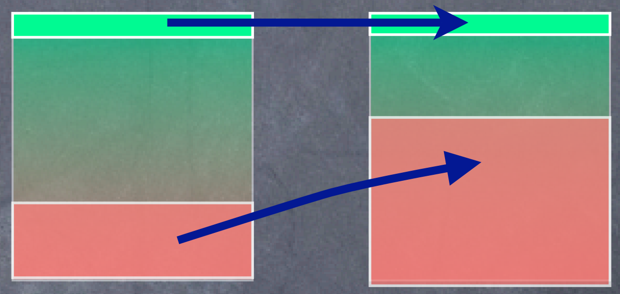

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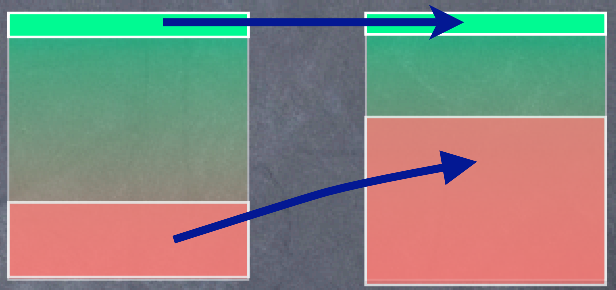

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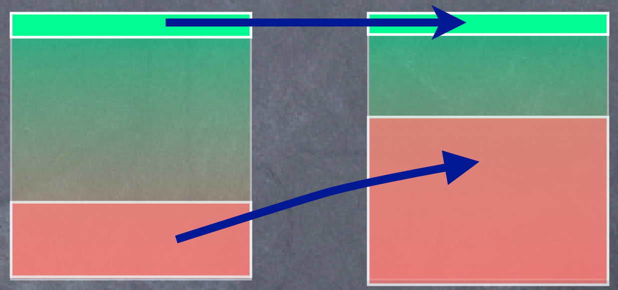

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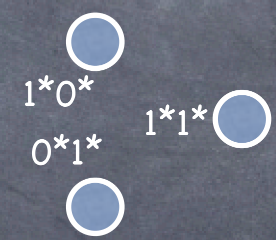

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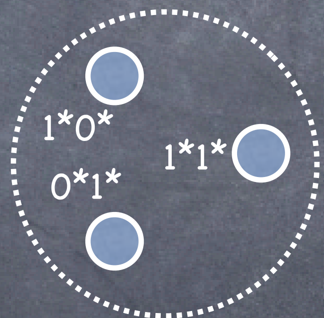

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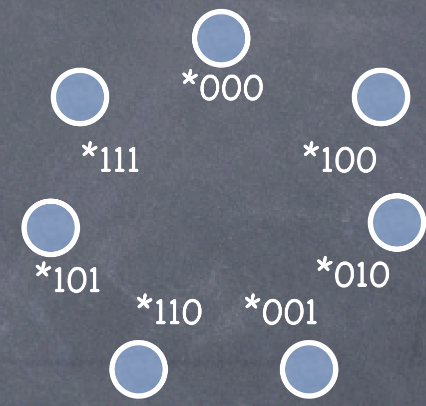

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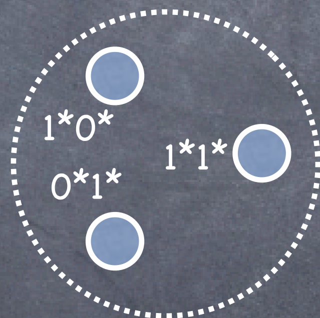

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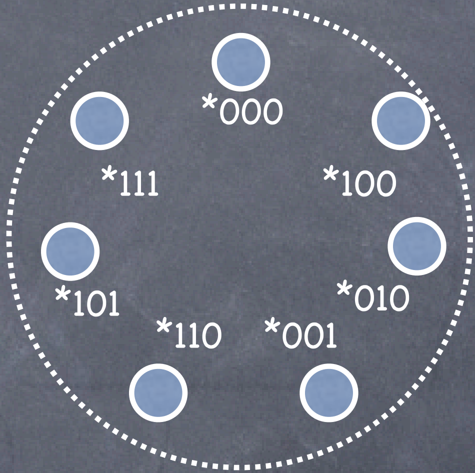

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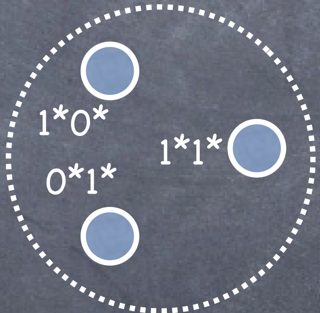

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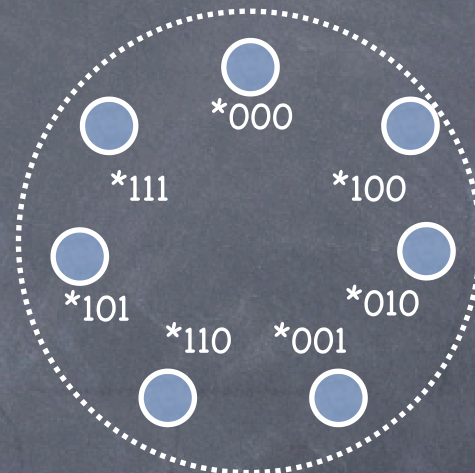

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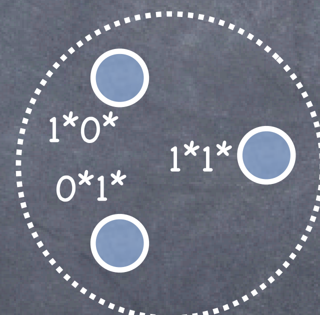

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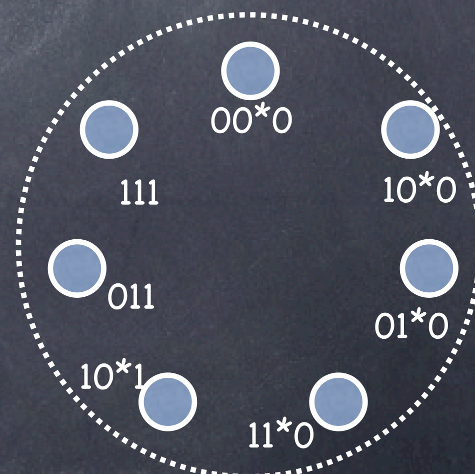

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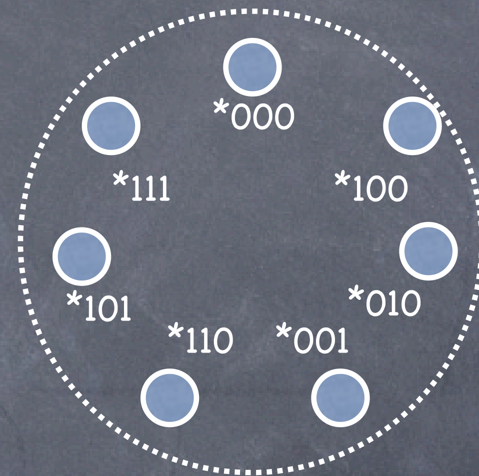

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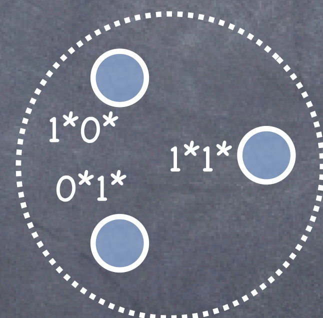

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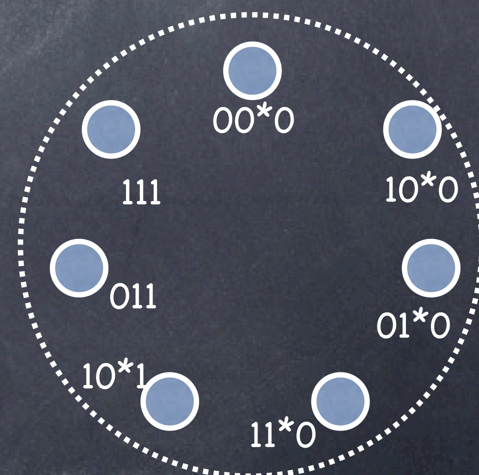

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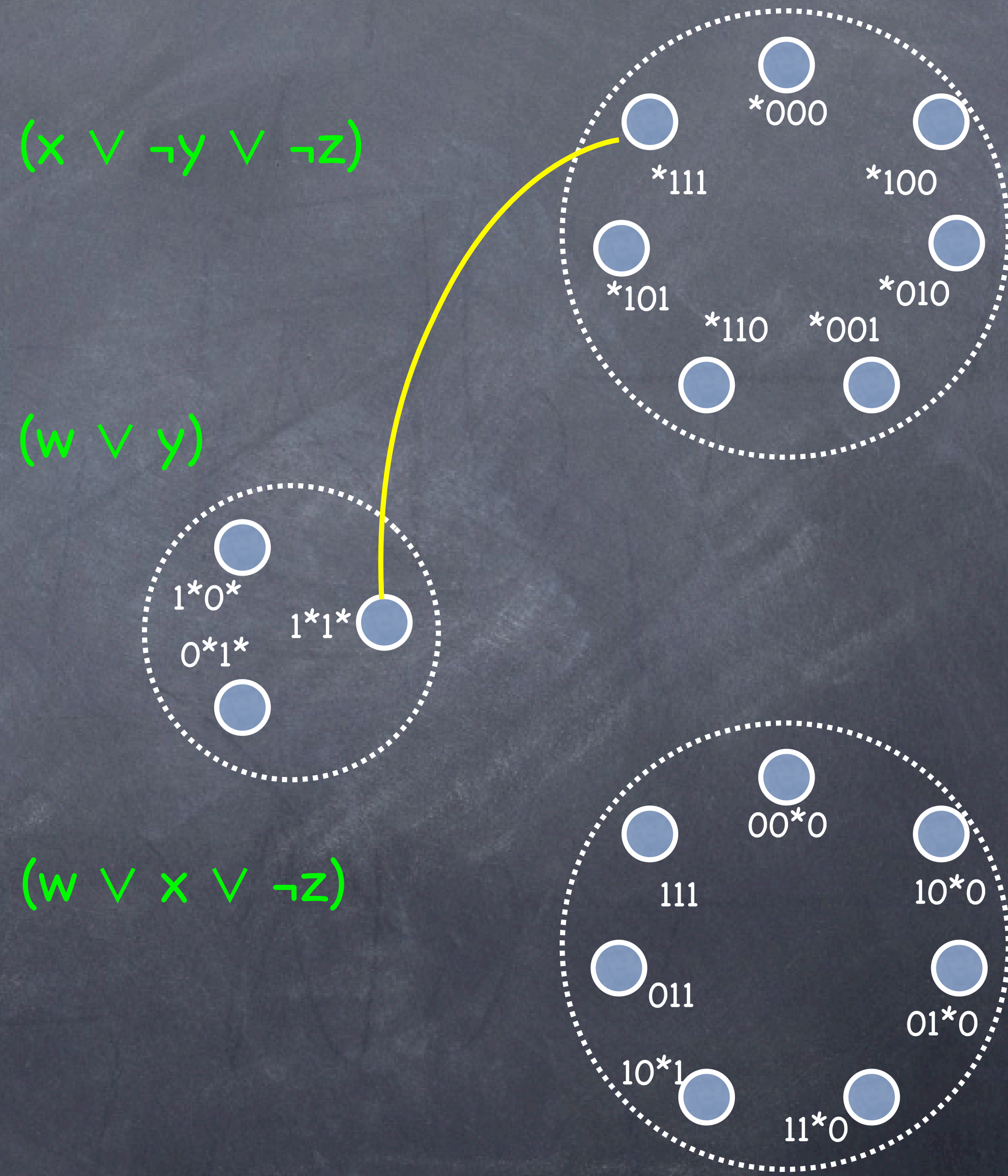

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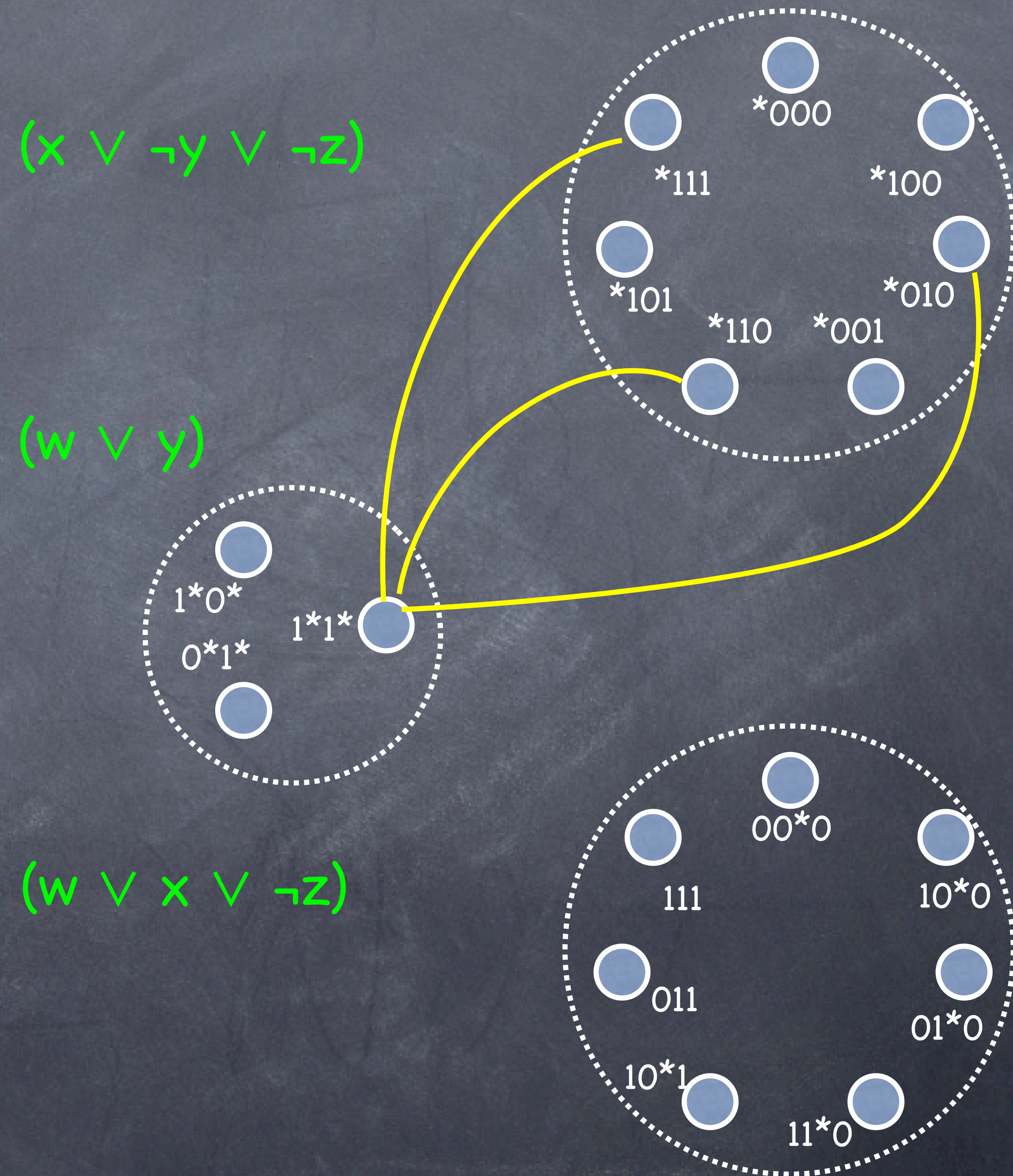

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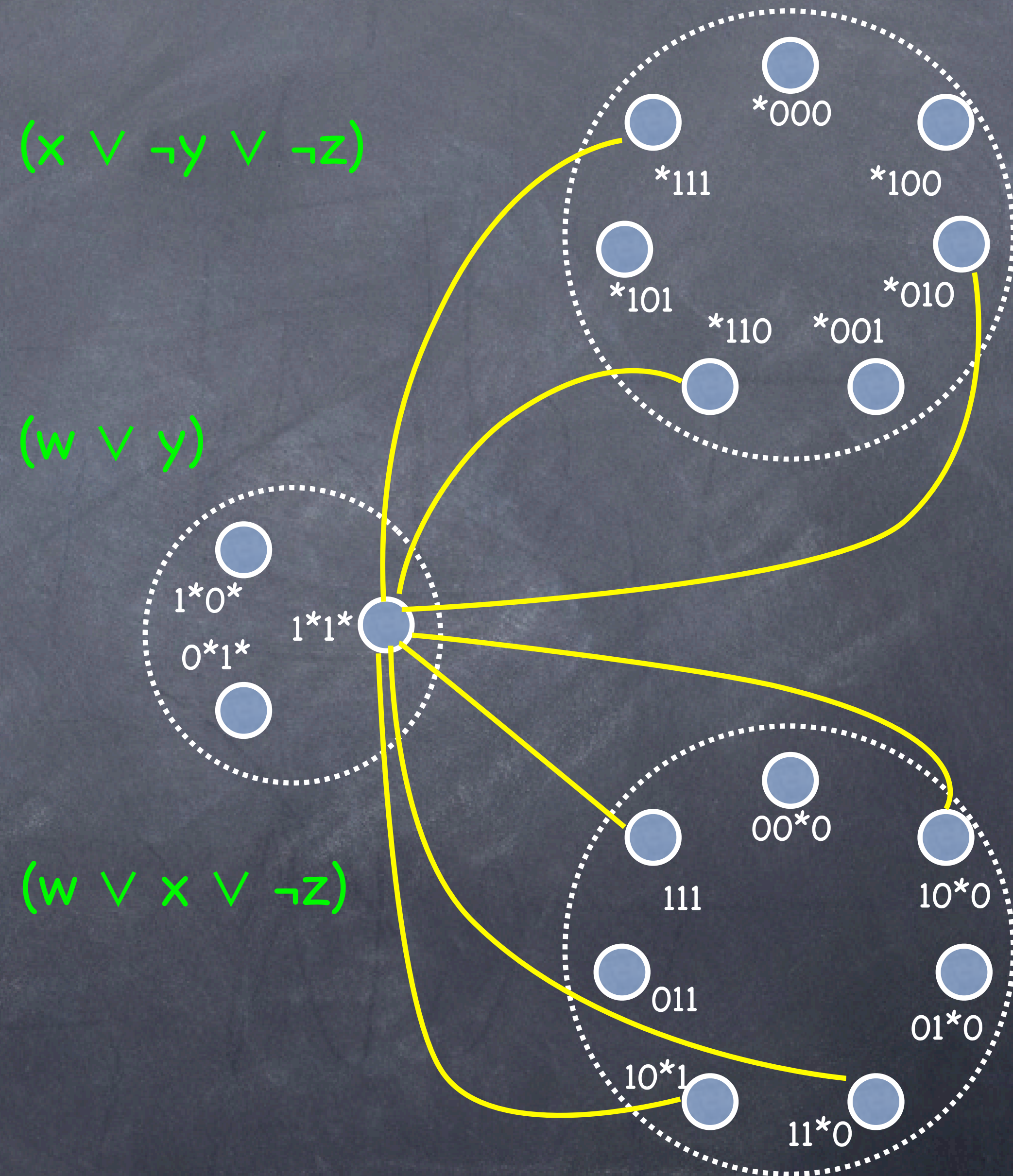

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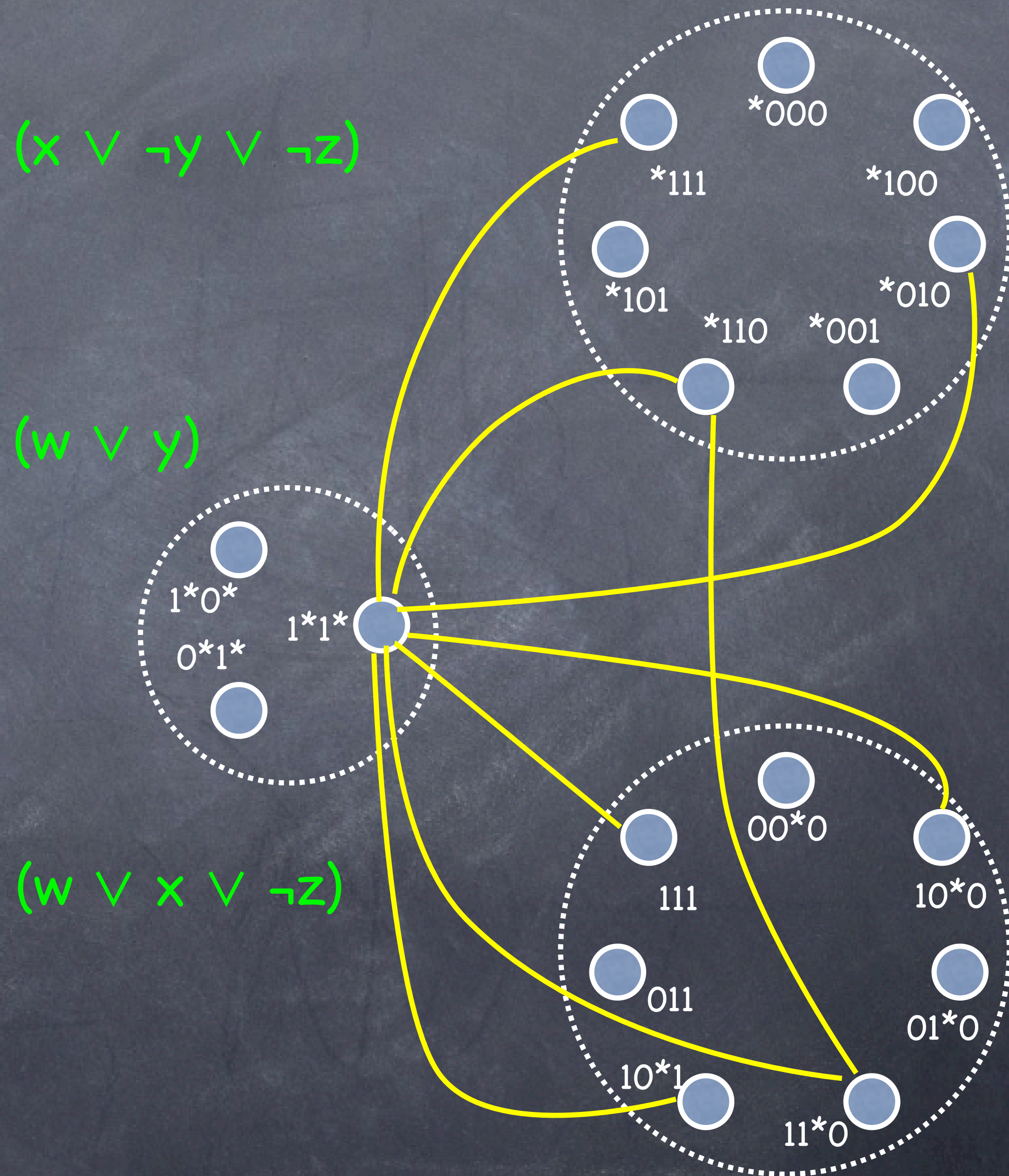

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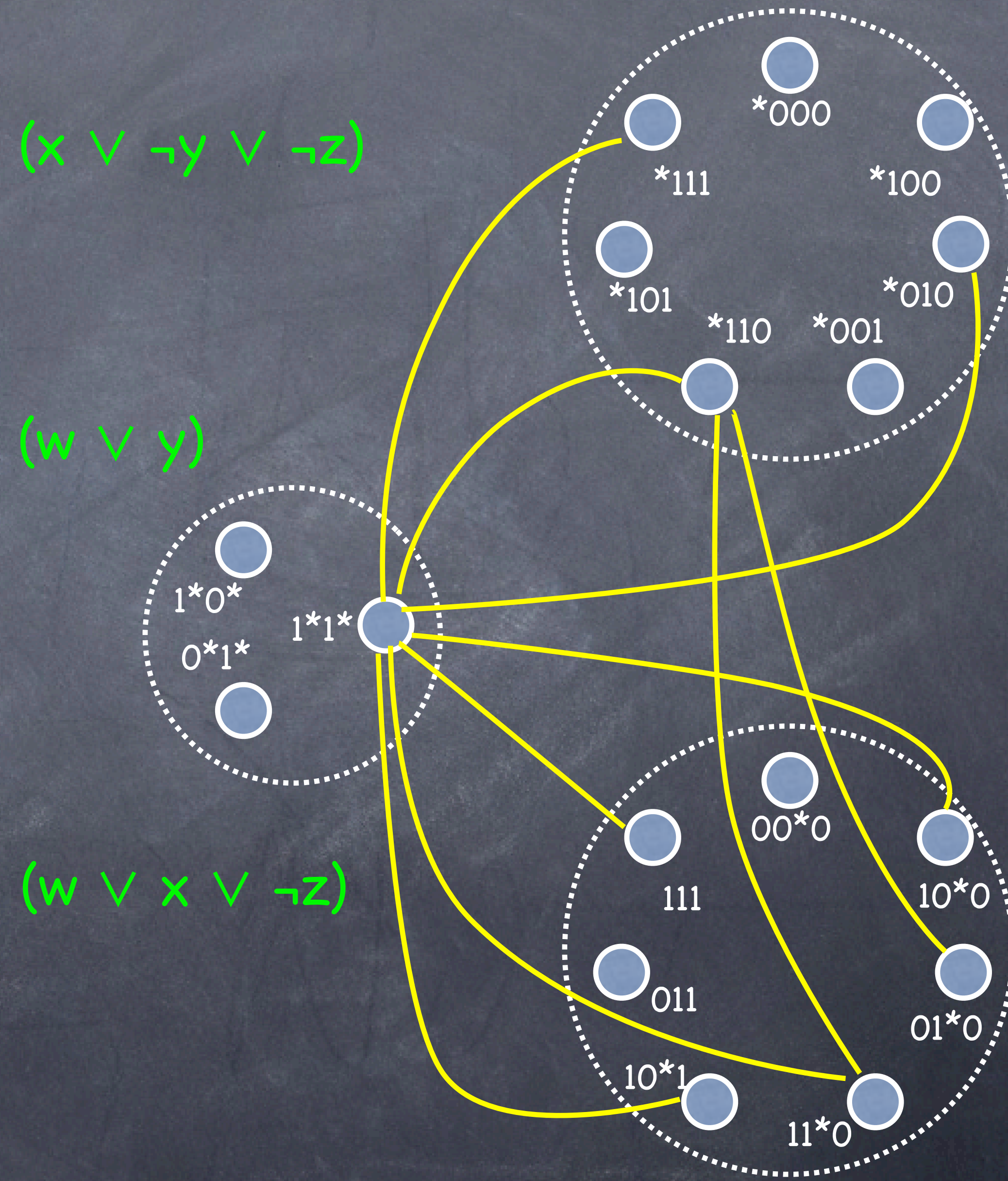

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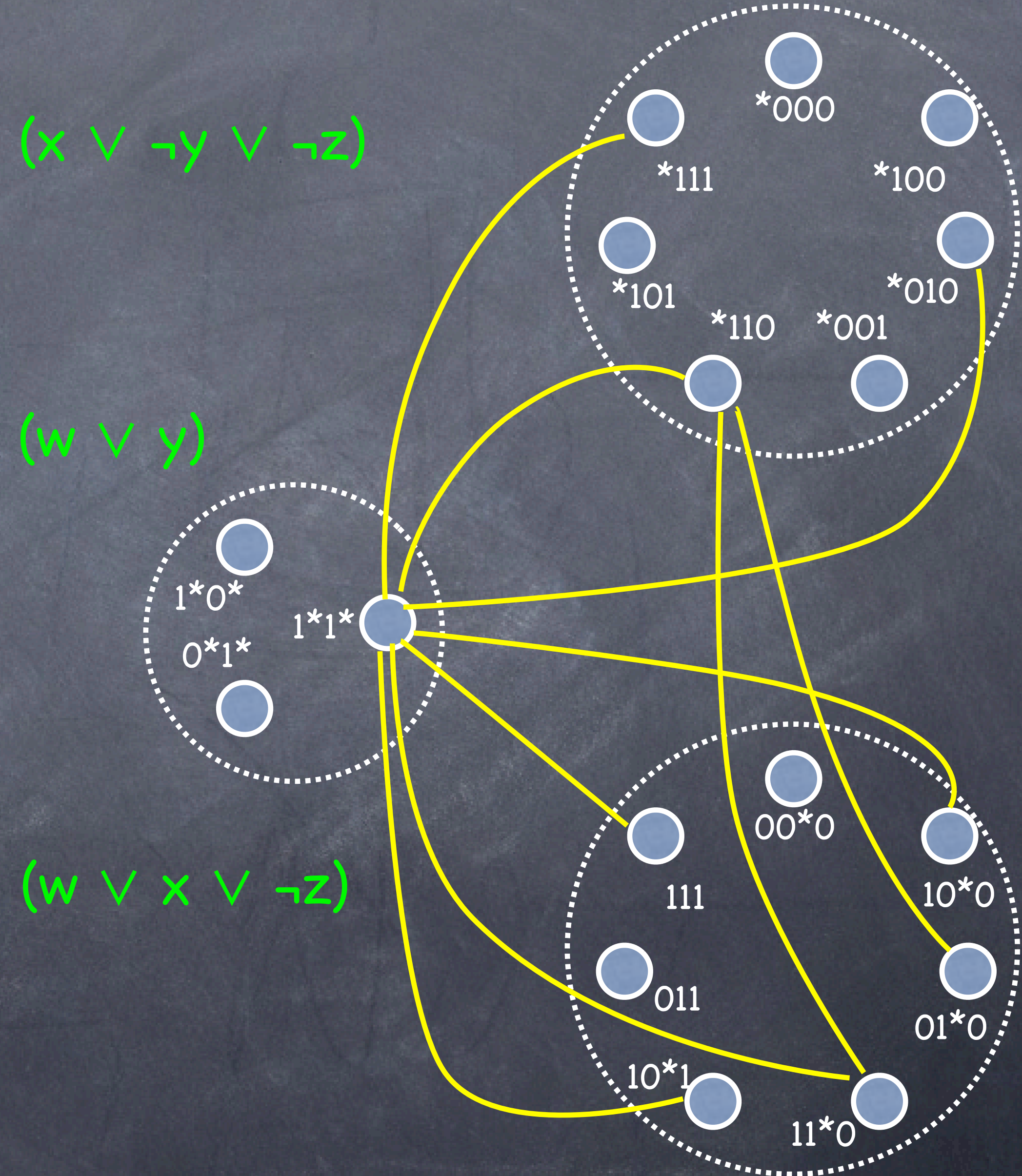

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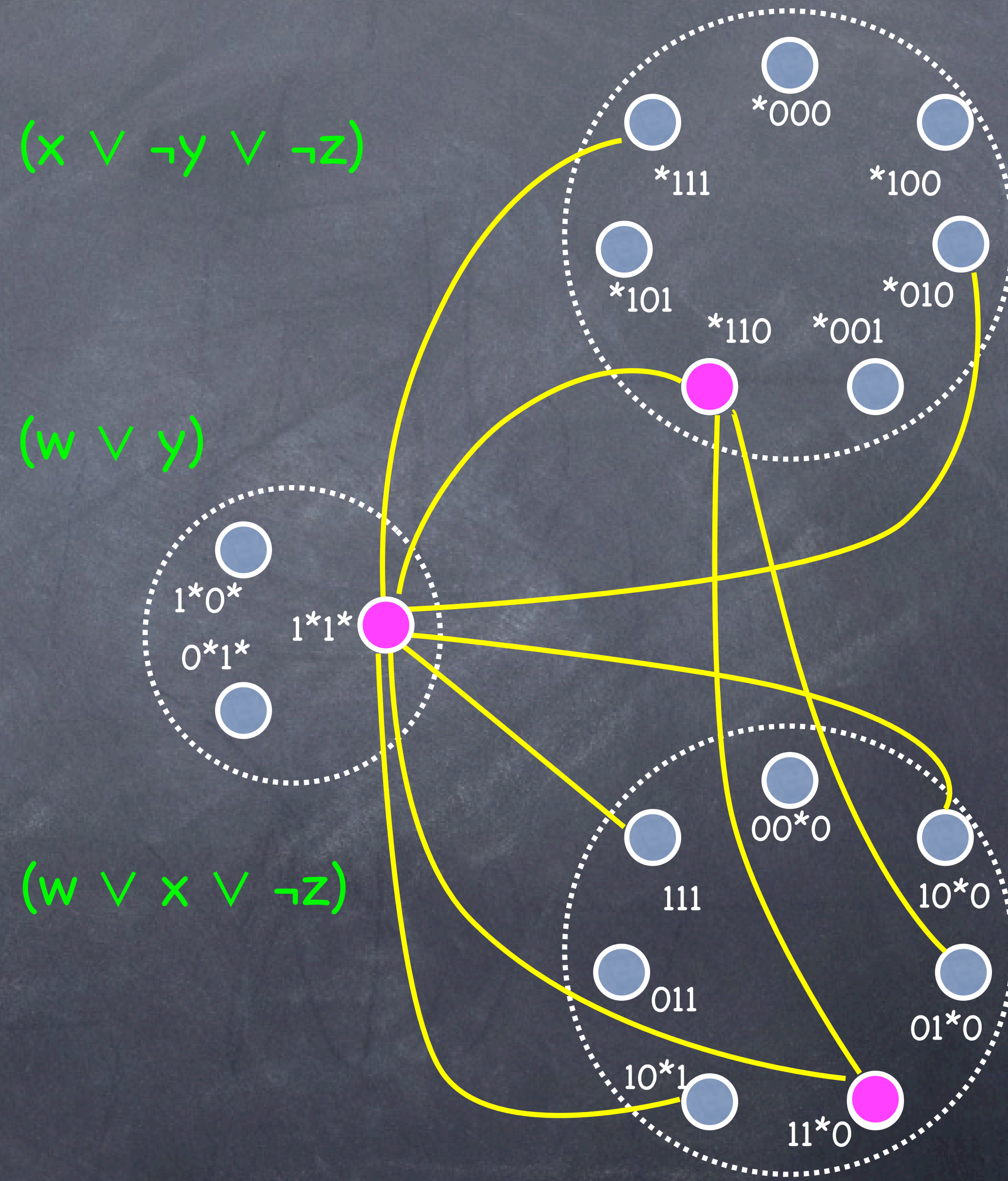

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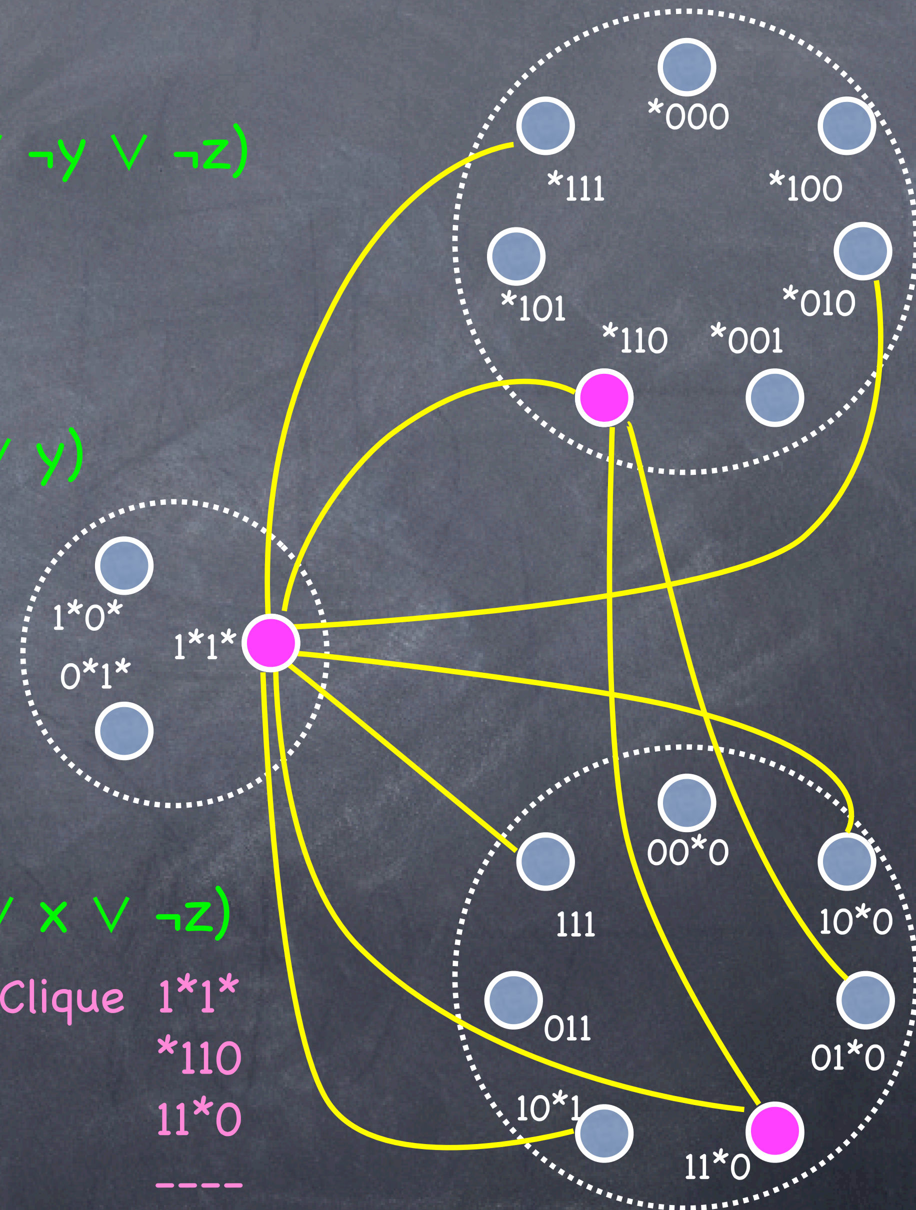

3-Clique 1*1*
*110
11*0
----
sat assignment 1110

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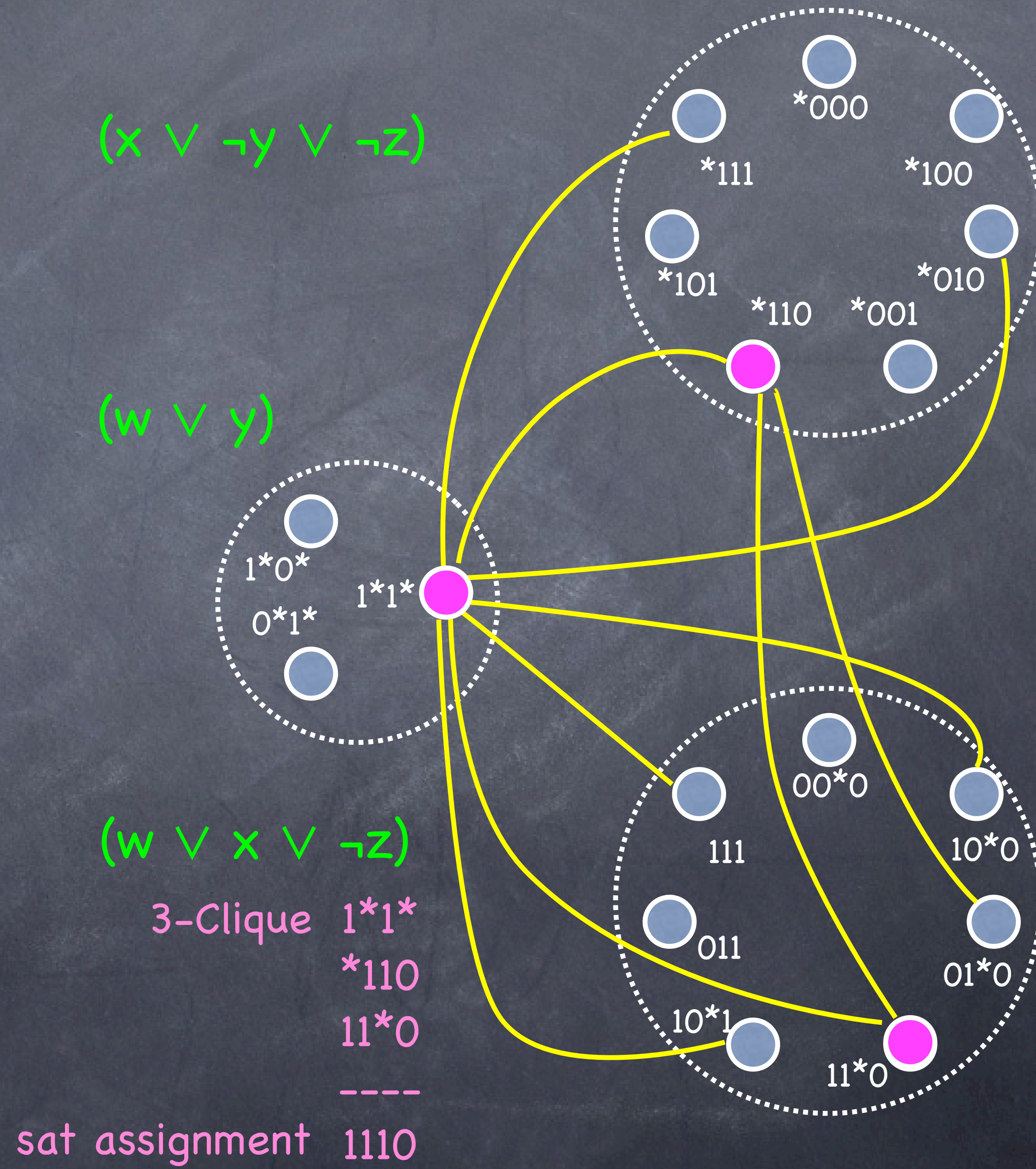

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  - Need to check that the encoded vector is the evaluation of the clauses on an assignment, and that encoding is valid

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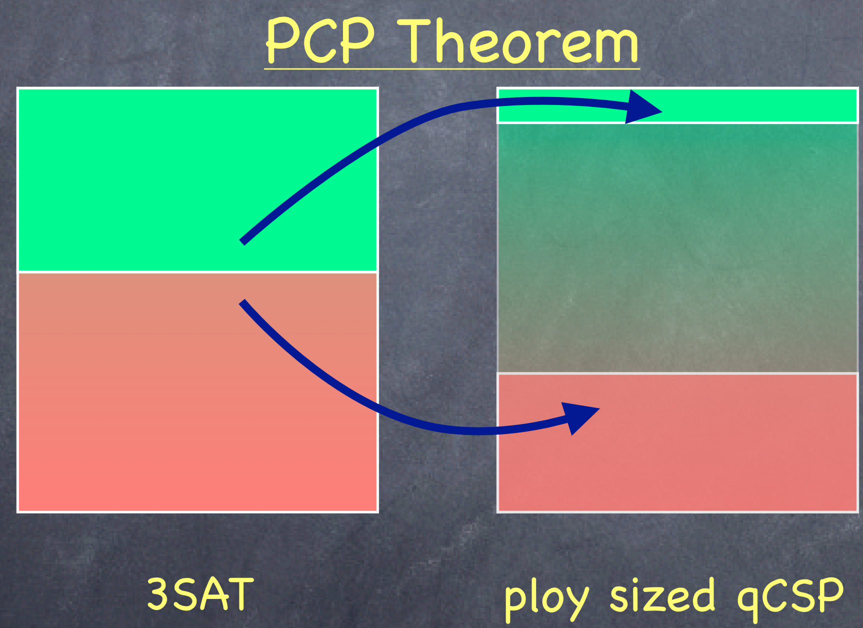

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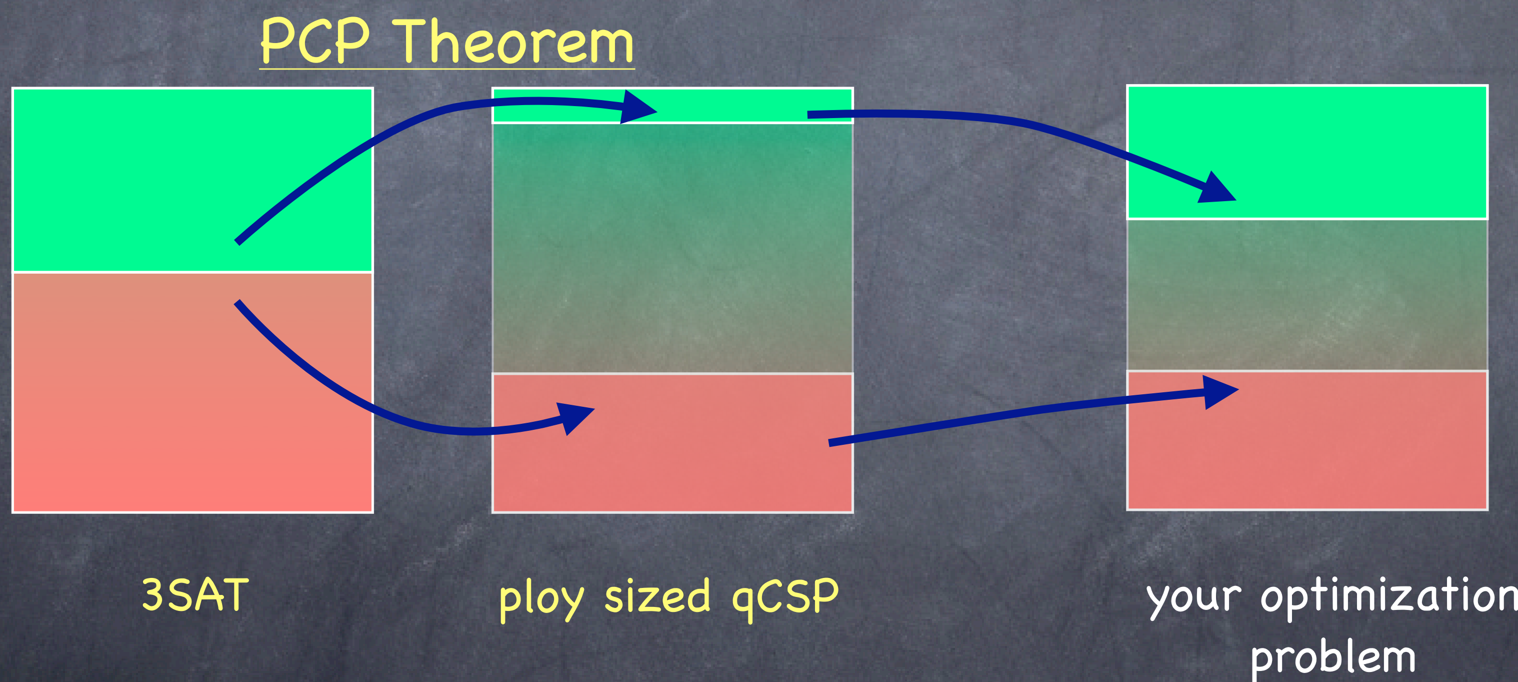

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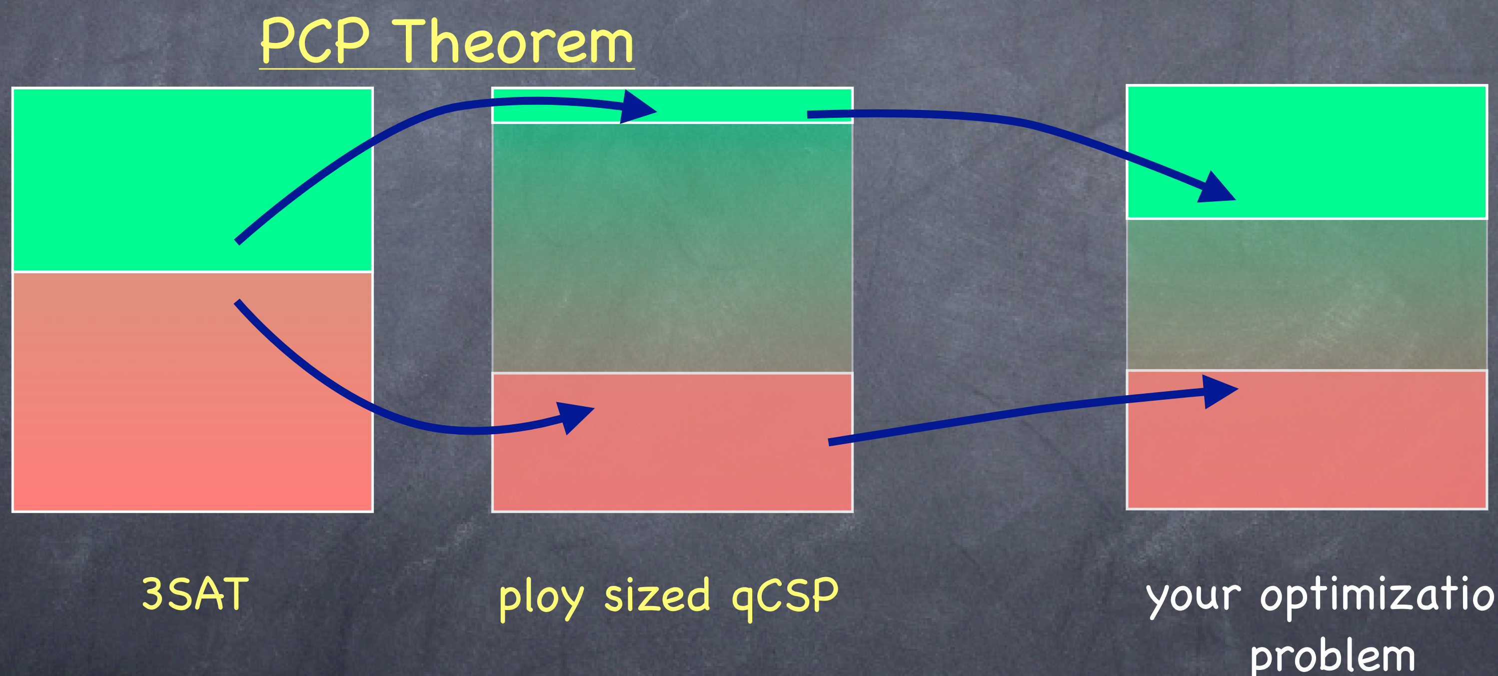

- Variants of these reductions to get different hardness results for different approximations