Decision Trees

Lecture 22
To left or to right

Decision Trees

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  - Number of bits of input read

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    - So even allow unbounded computational power
    - Simpler combinatorial structure (need not understand P vs. NP etc.)

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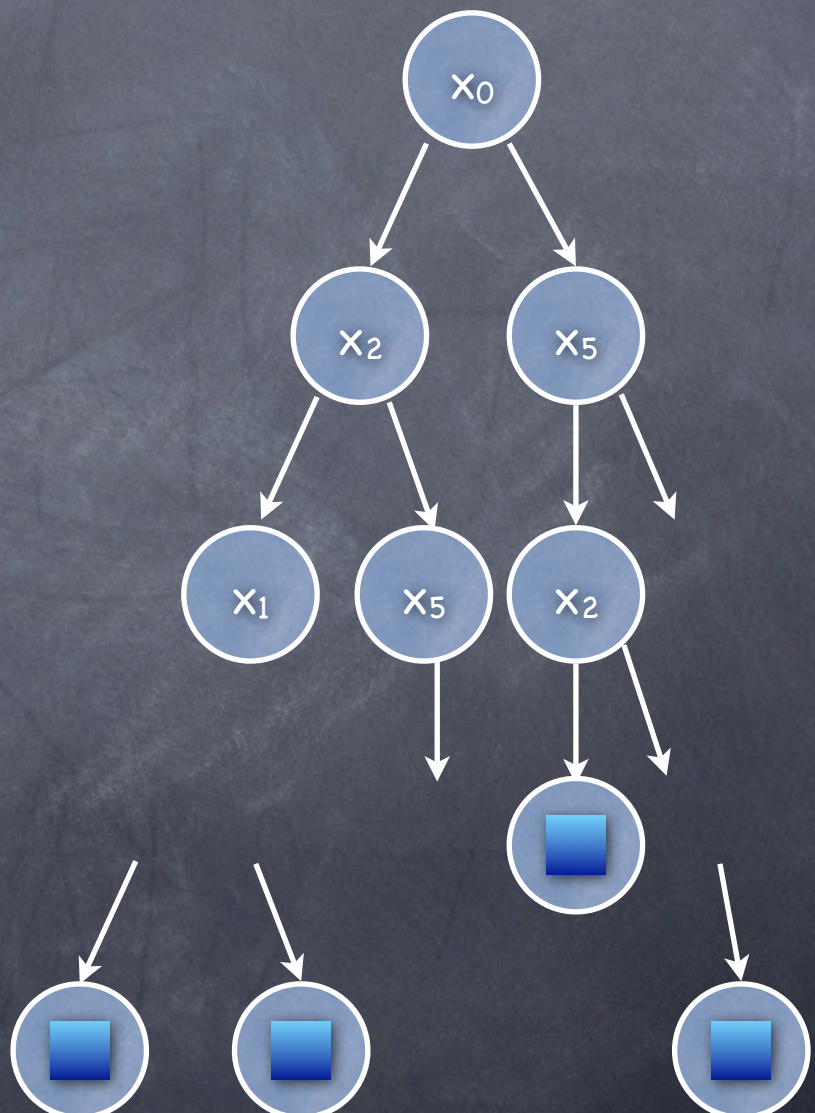

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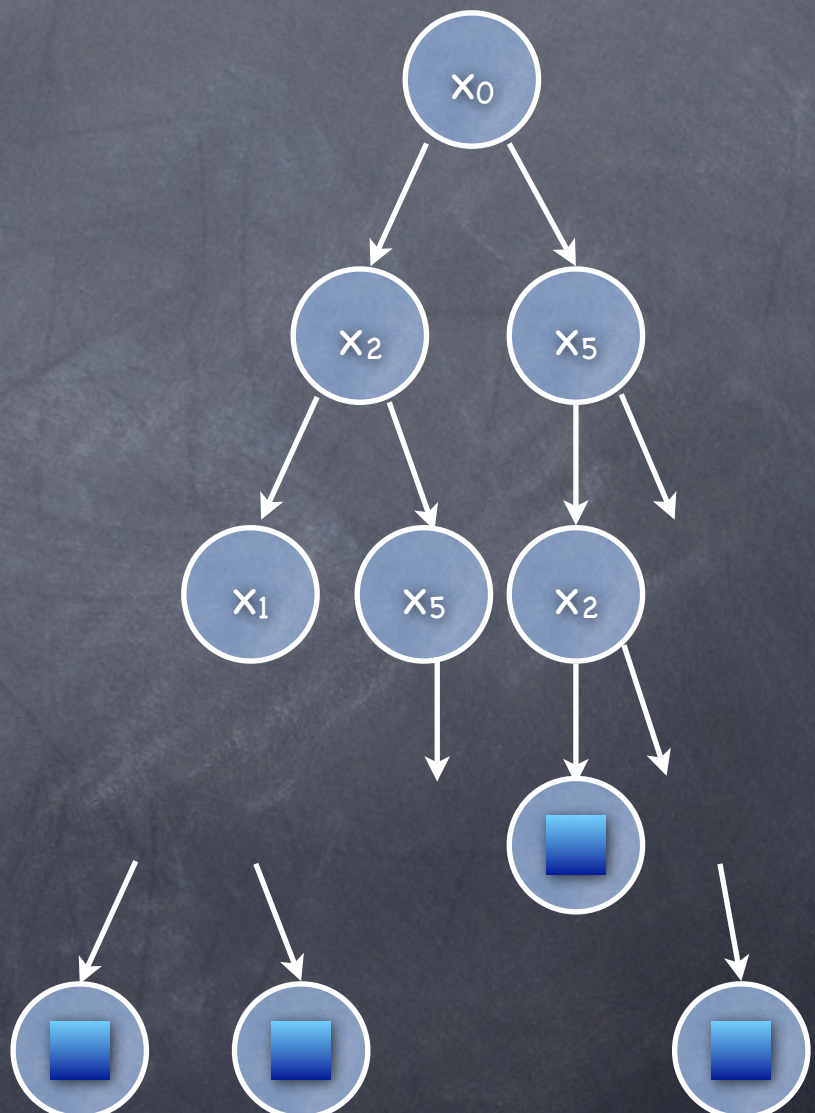

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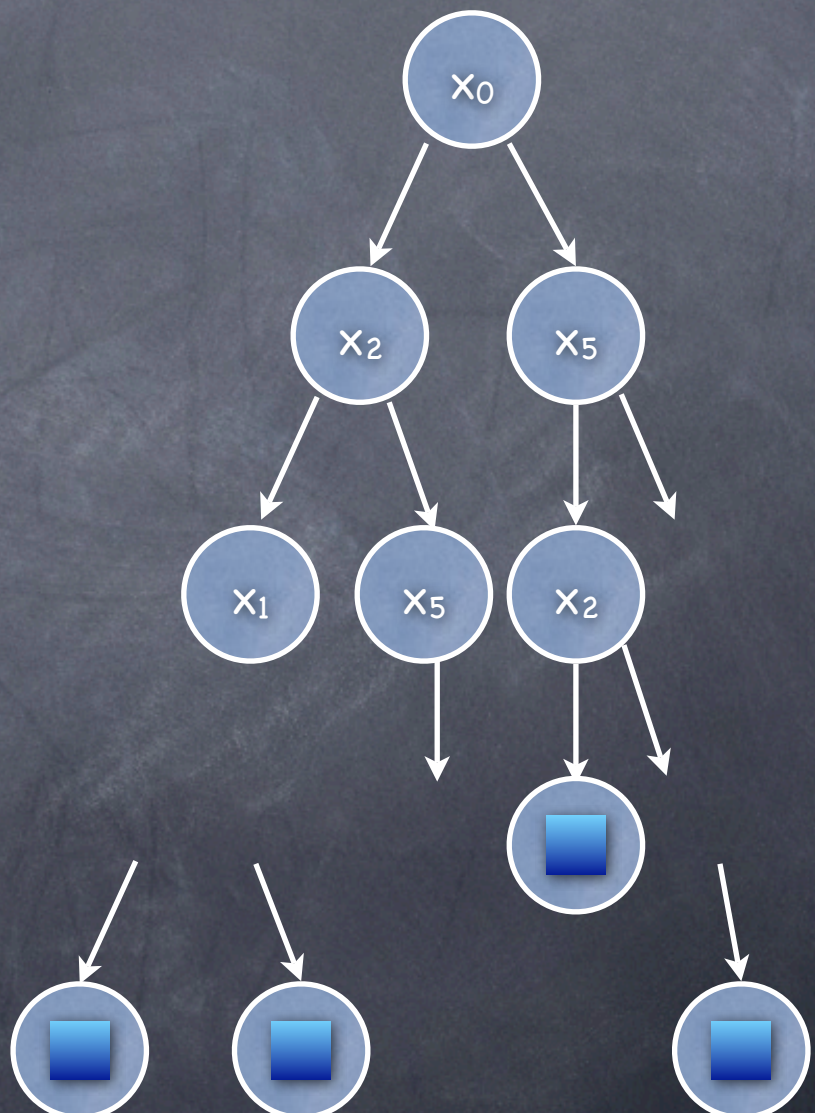

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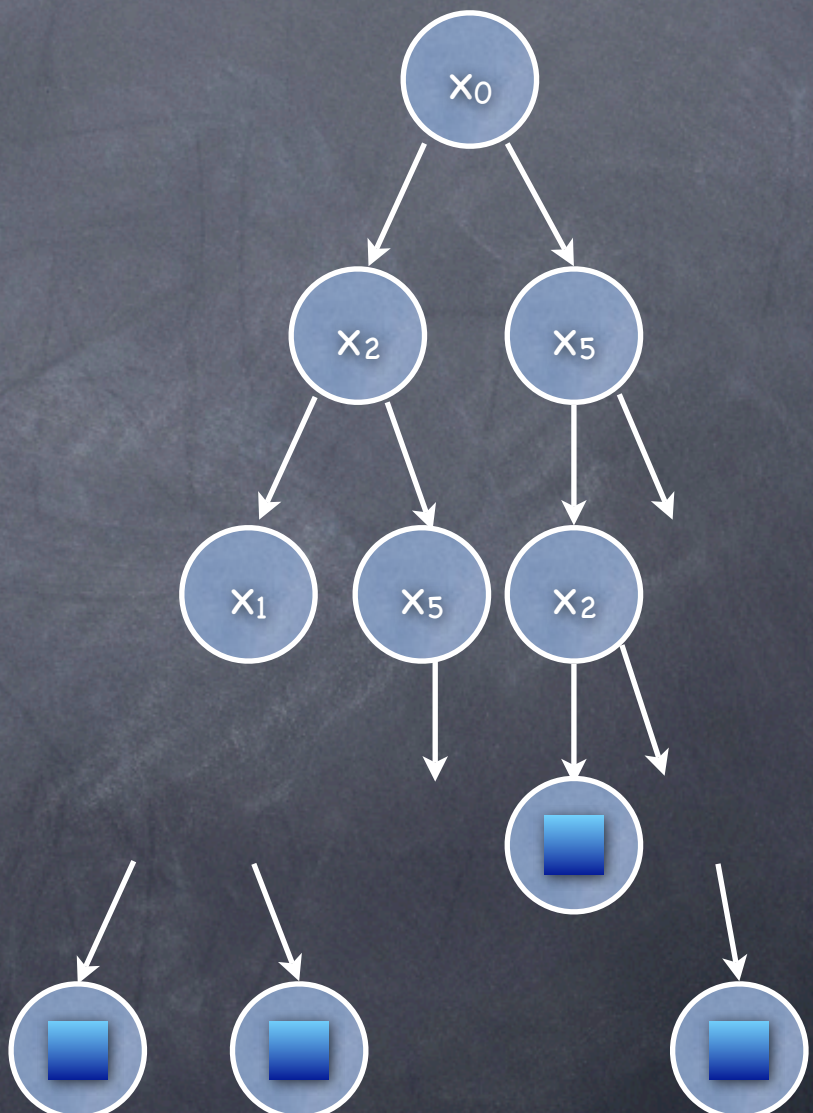
- Configuration graph of a computation, as it reads each bit
  - For n-bit input, depth at most n
  - Some paths may be shorter
- DTree(L) = $\min_{\text{alg A}} \max_{\text{input x}} T_{A,x}$ where $T_{A,x}$ is the number of bits of x read by A



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  - $SAT_C(x)$ if x is a satisfying assignment for circuit (or circuit family) C
  - CONNECTED(G) = 1 if G is the adjacency matrix of a connected graph
- We are interested in showing DTree lower-bounds for these problems

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    - Before n nodes, set of inputs contain 0^n and another input, no matter what bits where queried at the nodes

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  - Until then, graph can be connected or disconnected: by setting all unqueried edges to Yes or all to No

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  - Maj(x) = 1 iff #1s in x > #0s (assume |x| odd)
    - Adversary strategy: alternately answer 0 and 1

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    - Exercise

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  - For x s.t. L(x)=1, a subset of the bits of x which proves that L(x)=1 : c s.t. $x|c \Rightarrow x \in L$ (i.e., no x′ s.t. L(x′)=0 and has the same values at those positions)

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$$DTree(L) \leq 0Cert(L) \times 1Cert(L)$$

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  - Clearly correct. Number of bits read?

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  - In each iteration at most 0Cert(L) bits queried

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  - 0-certificate: enough variables so that can evaluate just one input wire for AND gates, and all input wires for OR gates
  - 1-certificate: enough variables so that can evaluate just one input wire for OR gates, and all input wires for AND gates
  - If regular AND-OR tree, 0Cert(L) x 1Cert(L) = number of leaves = DTree(L)

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  - "Sensitivity" is a lower-bound on DTree(L)
- Will explore some in exercises

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  - Flip all coins up front and then run a deterministic computation
    - i.e., randomly choose a (deterministic) decision tree

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- Question: How to prove lower-bounds against randomization?

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	(Deterministic) Algorithms			
Inputs				
		$T_{A,x}$		

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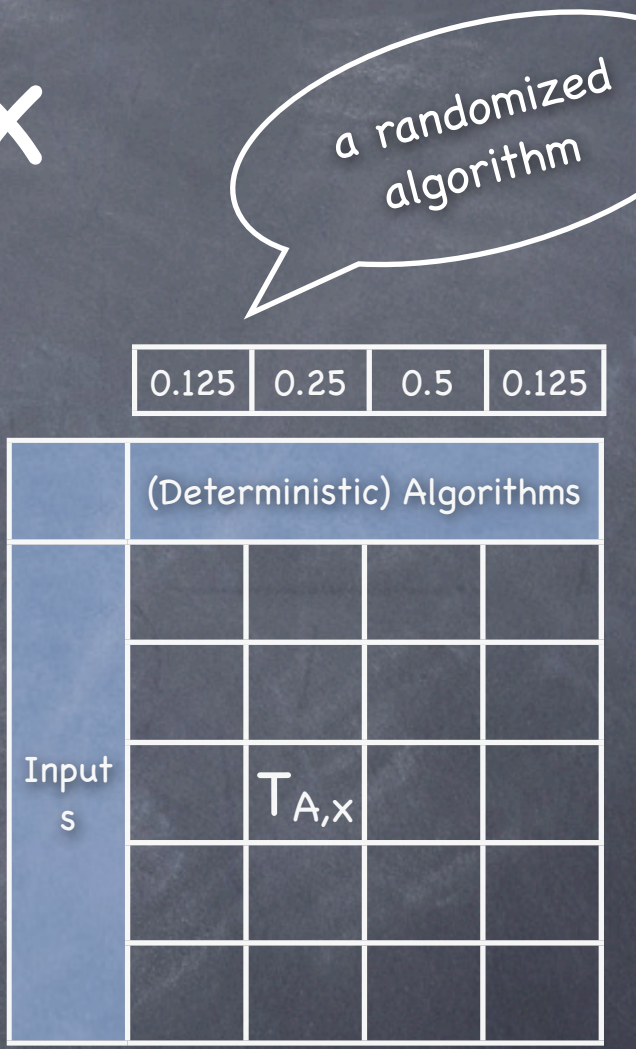

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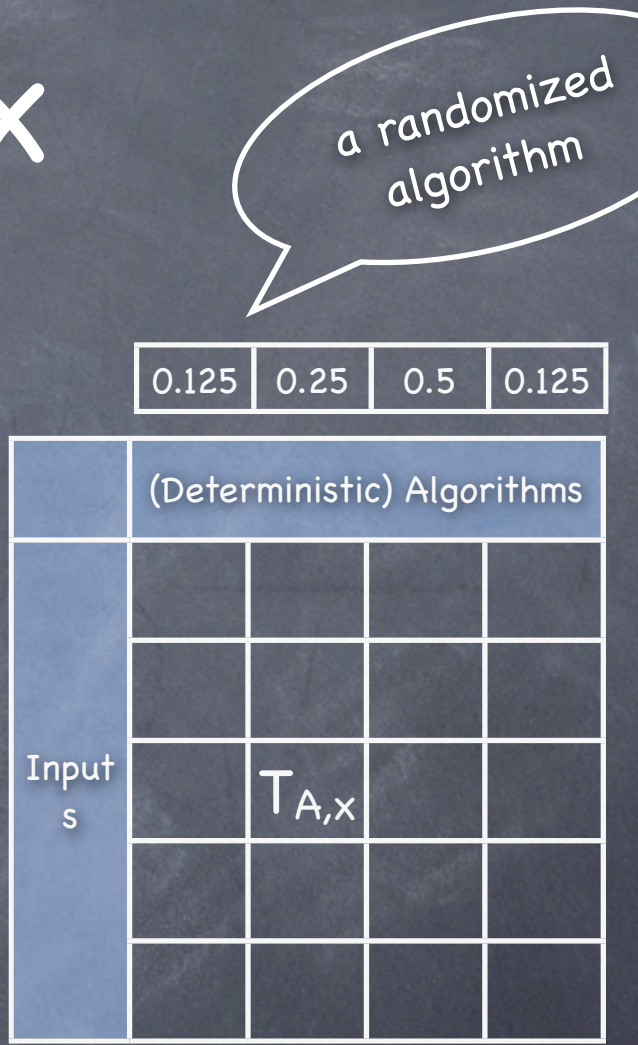

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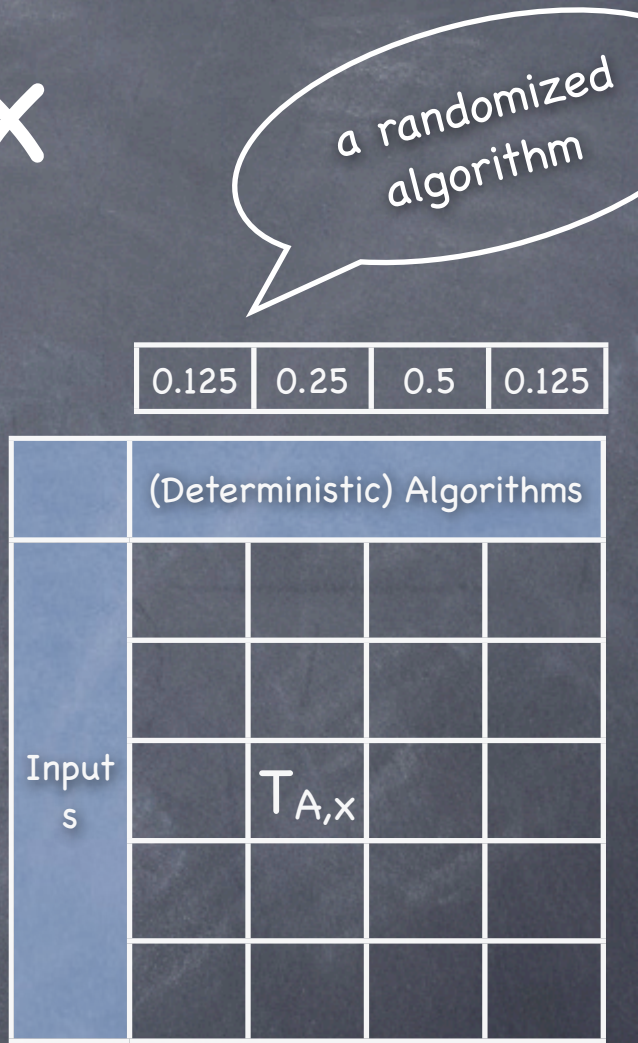

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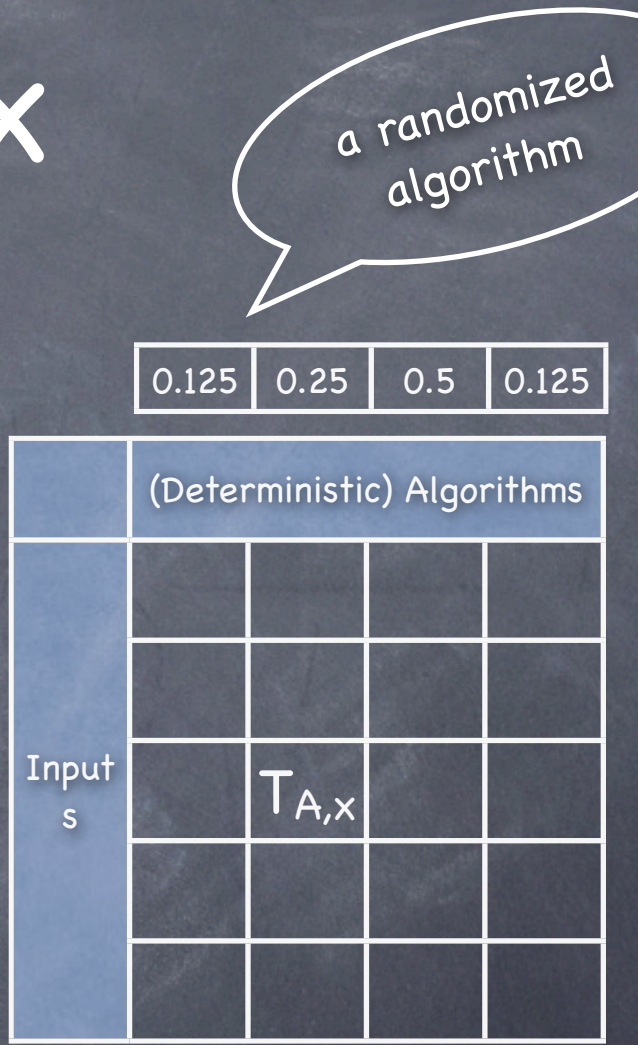

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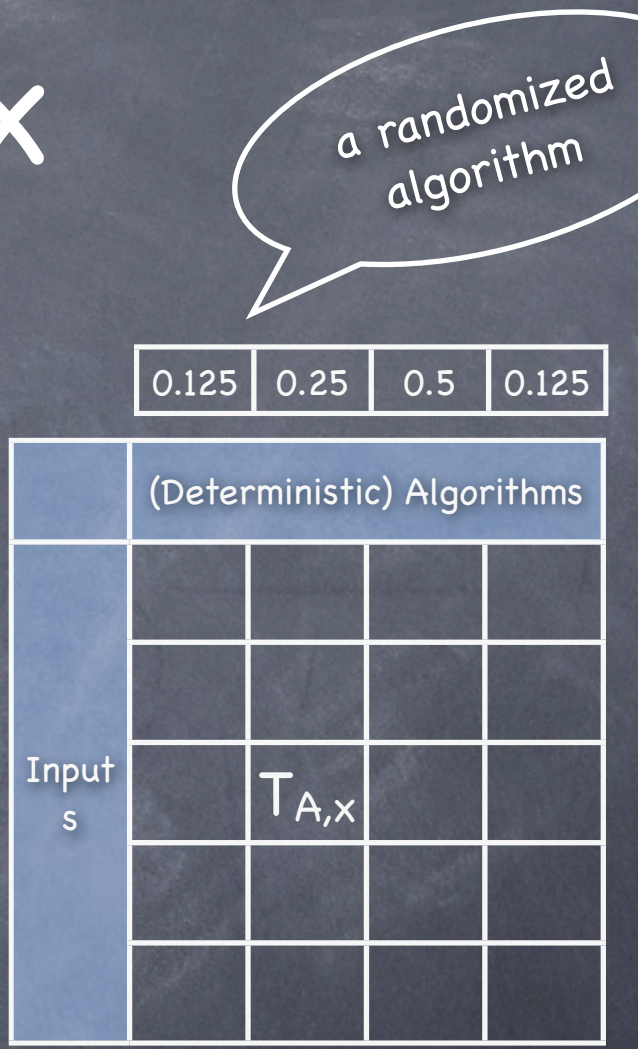

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- Another setting: Given adversarial input distribution X; pick your deterministic algorithm A
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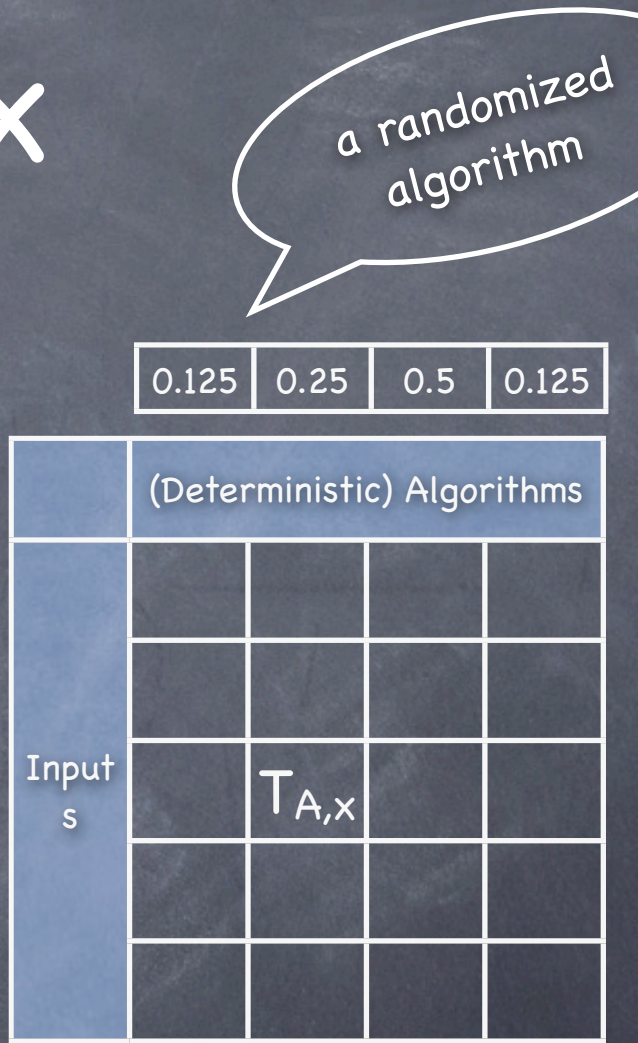
- Both have the same expected cost!! (not obvious: follows from LP duality)



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  - If every algorithm A performs badly on an **<u>input-distribution</u>** X, then a randomized combination of those algorithms also perform badly on X. If R does badly on X, on some x in its support it does at least as badly (x depends on R)

Yao's Min-Max

- $\min_{\text{rand-alg } R} \max_{\text{input } x} E_{A \leftarrow R}[T_{A,x}] = \max_{\text{inp-distr } X} \min_{\text{alg } A} E_{x \leftarrow X}[T_{A,x}]$
- Simpler, but useful direction: for any randomized alg R and any input-distribution X, $\max_{\text{input } x} E_{A \leftarrow R}[T_{A,x}] \geq \min_{\text{alg } A} E_{x \leftarrow X}[T_{A,x}]$
  - If every algorithm A performs badly on an **input-distribution** X, then a randomized combination of those algorithms also perform badly on X. If R does badly on X, on some x in its support it does at least as badly (x depends on R)
  - Useful: Can show lower-bound for randomized algorithms via lower-bound on distributional complexity for deterministic algorithms