Interactive Proofs

Lecture 17
IP = PSPACE

So far

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- AM, MA

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  - In fact, IP[k] in AM[k+2]

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- PSPACE $\subseteq$ IP
  - Prover can convince verifier of high complexity statements

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    - Or modify the proof (as we'll do)

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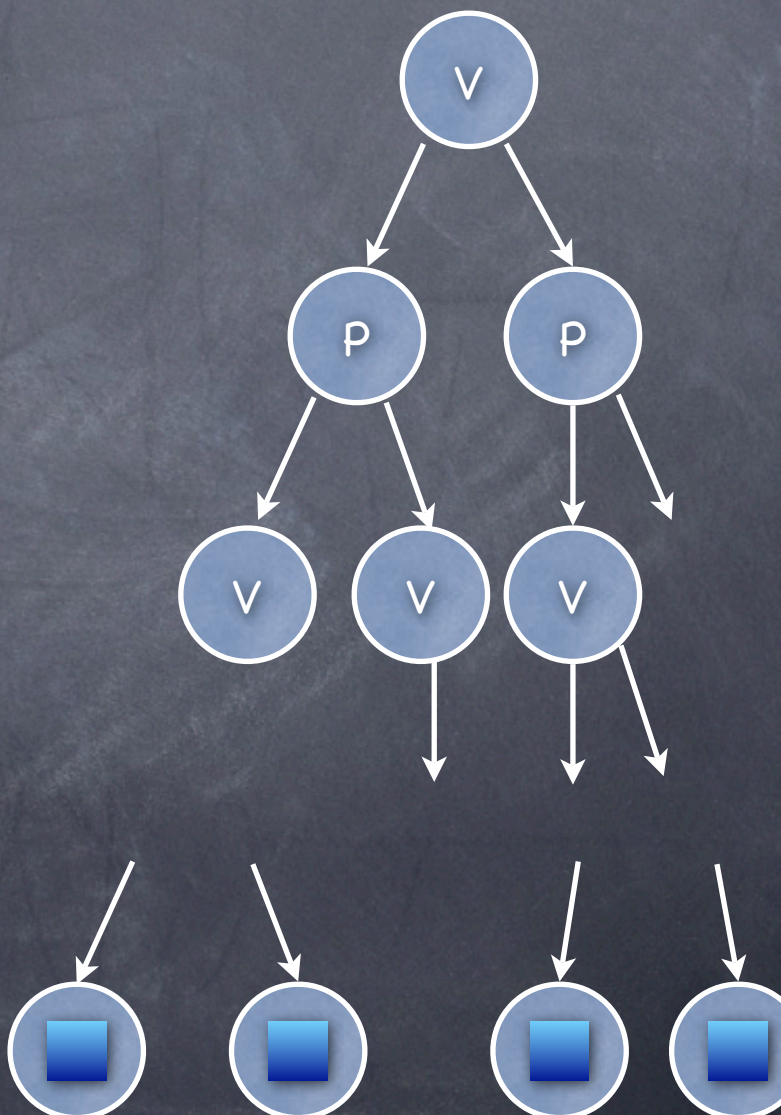

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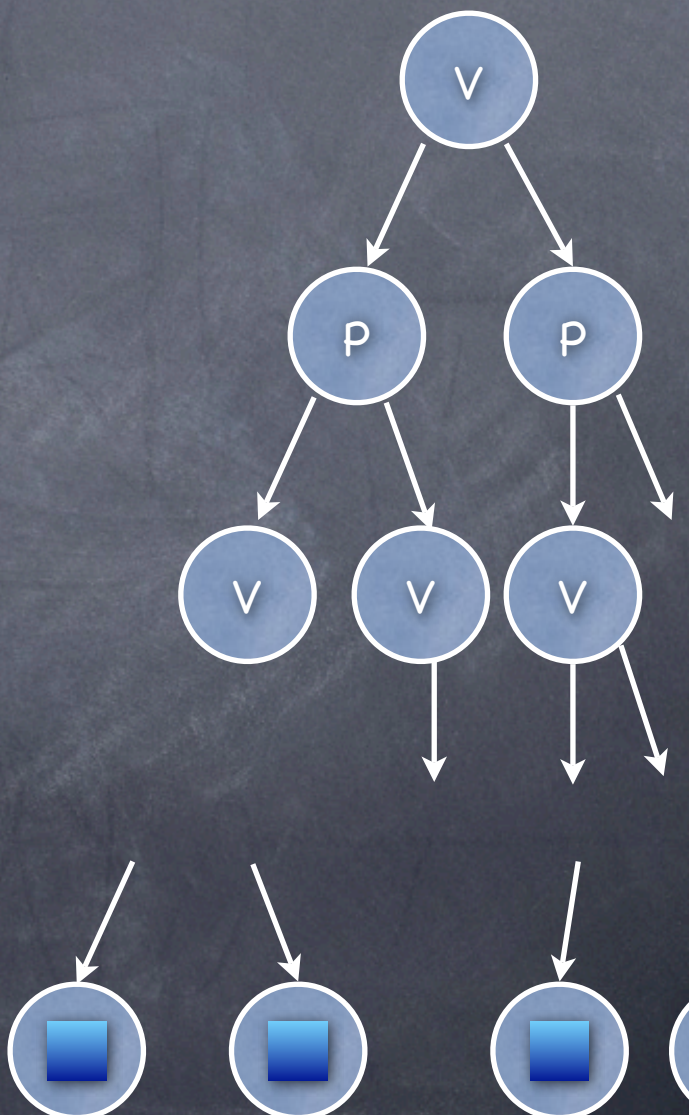

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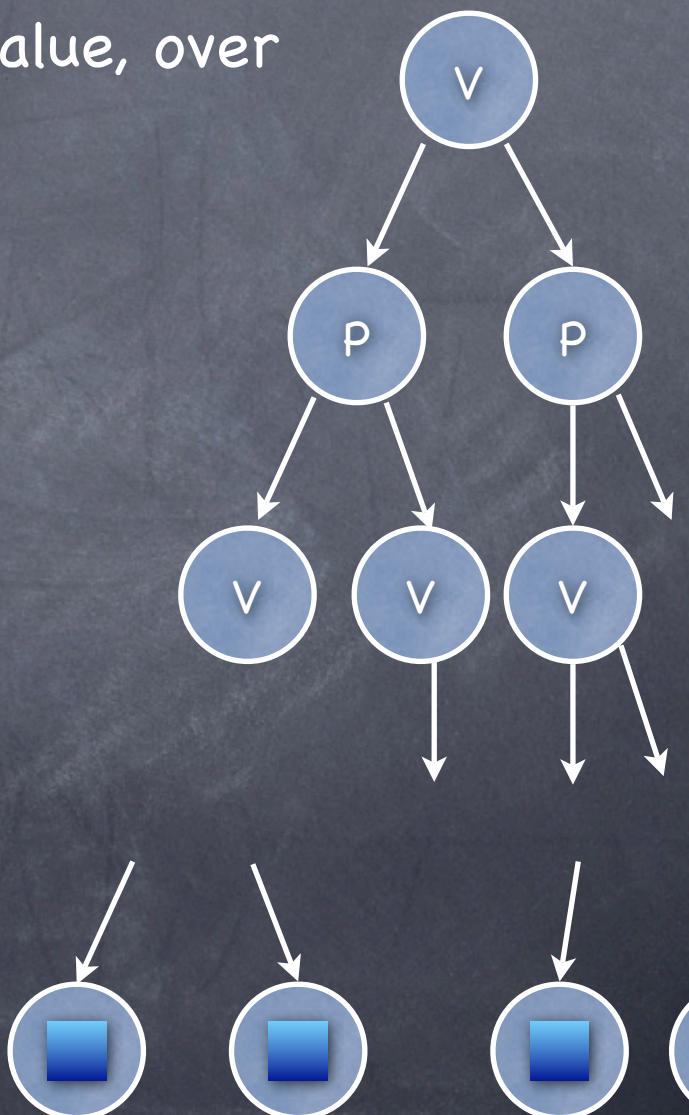

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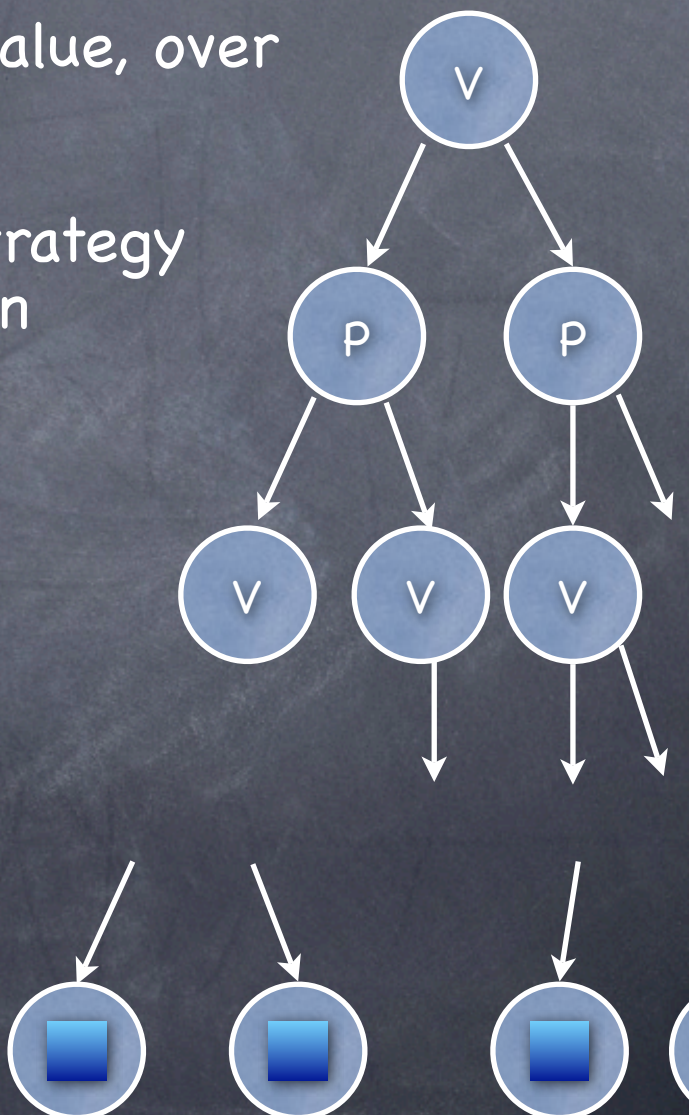

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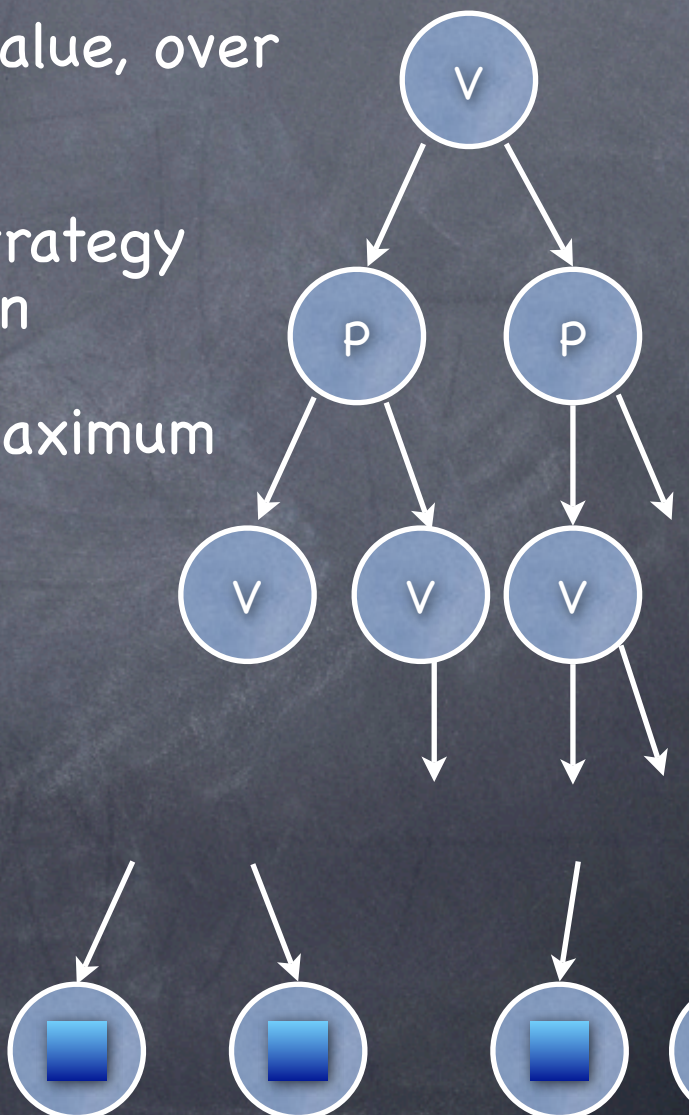

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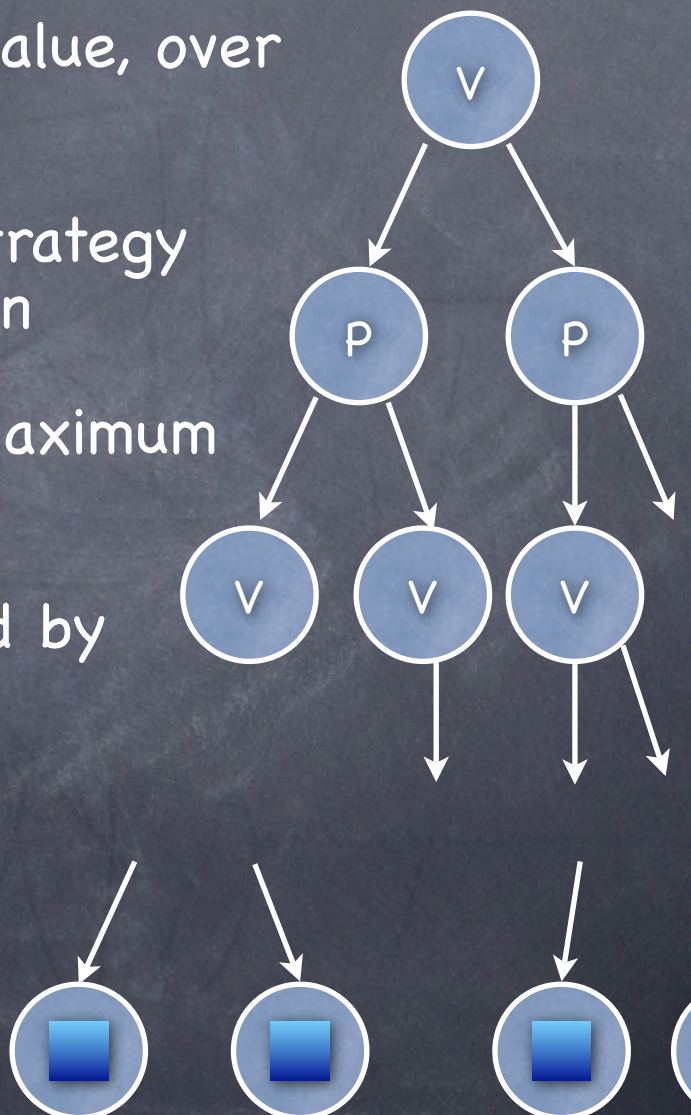

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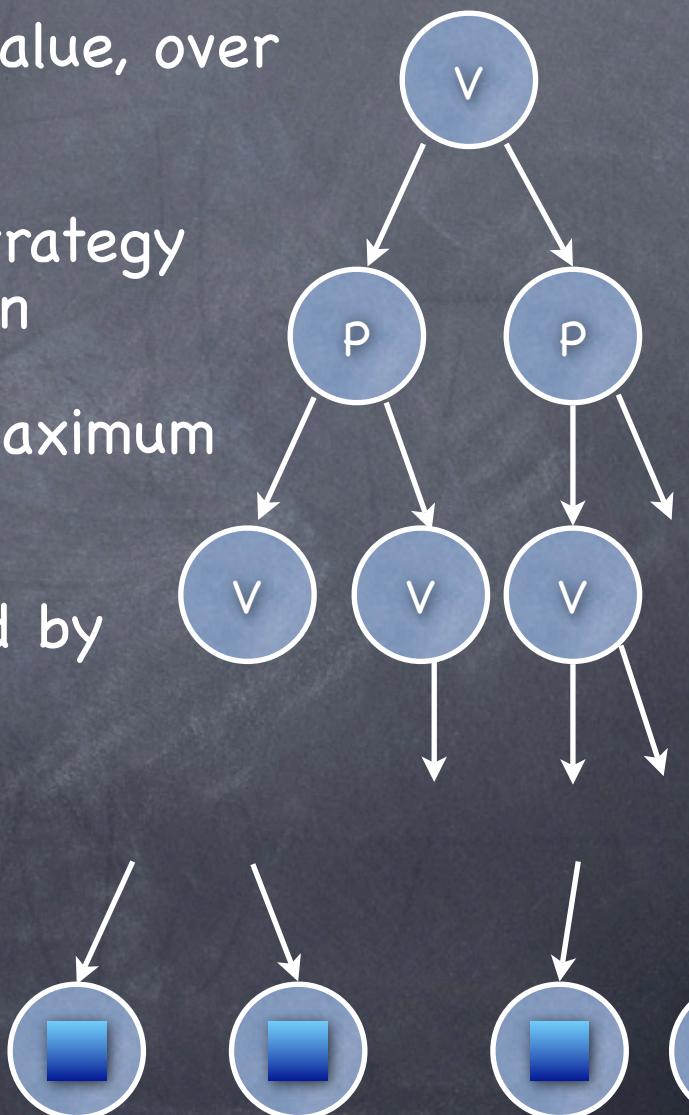

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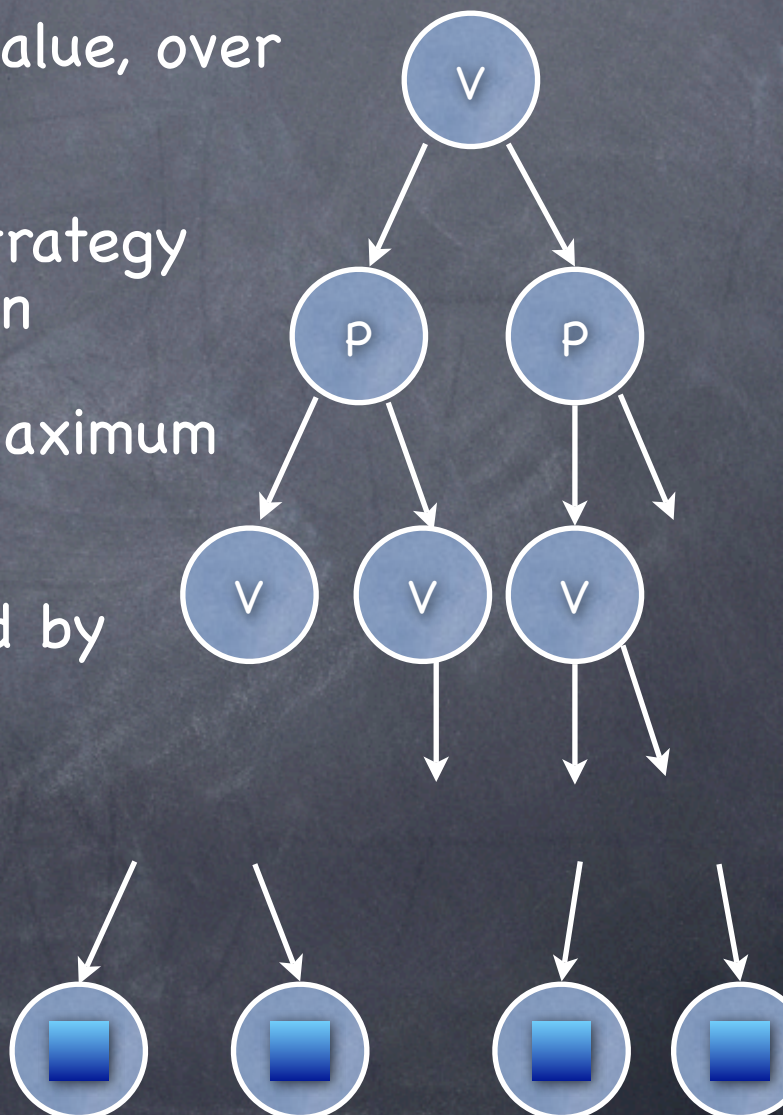

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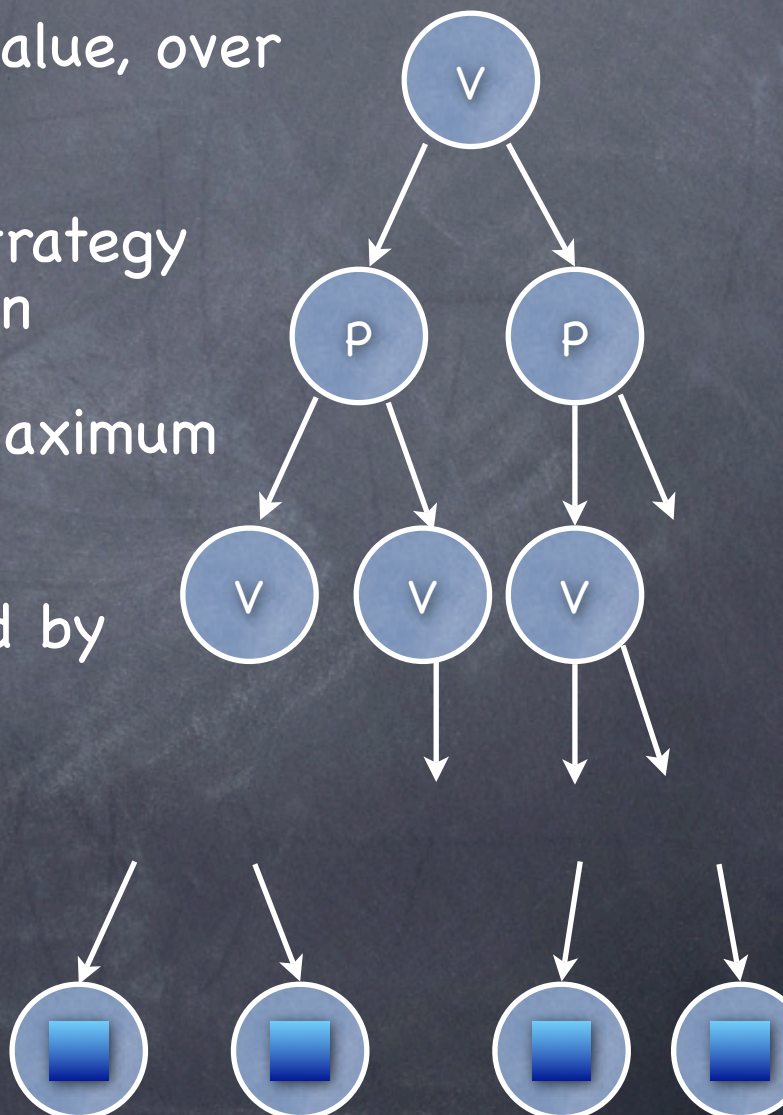

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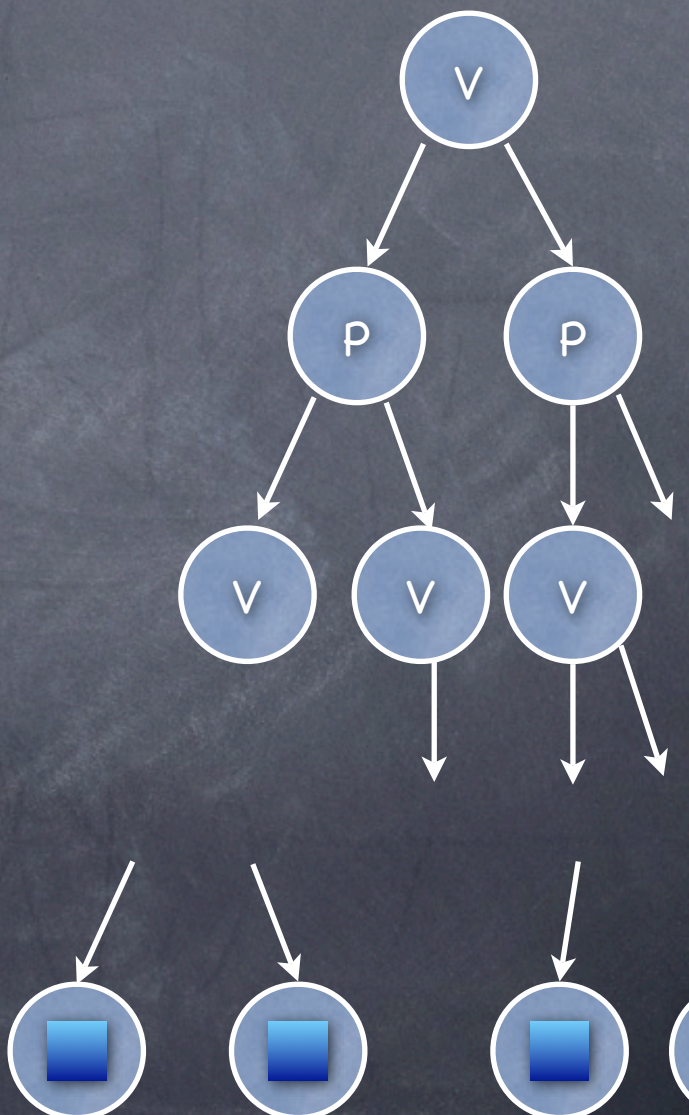

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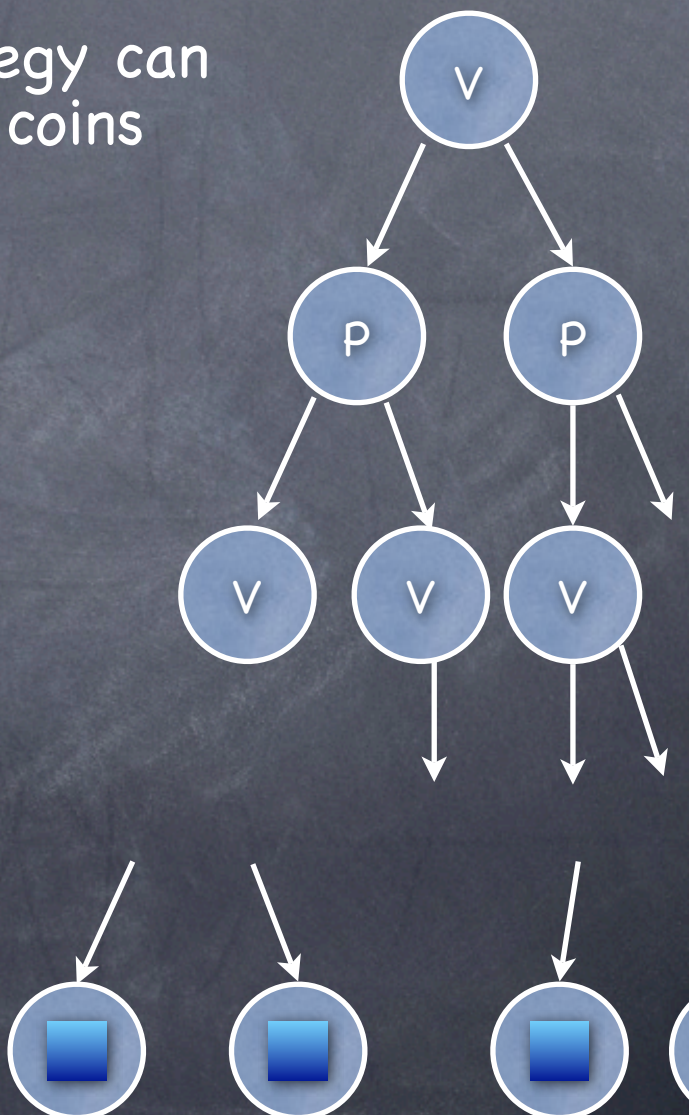

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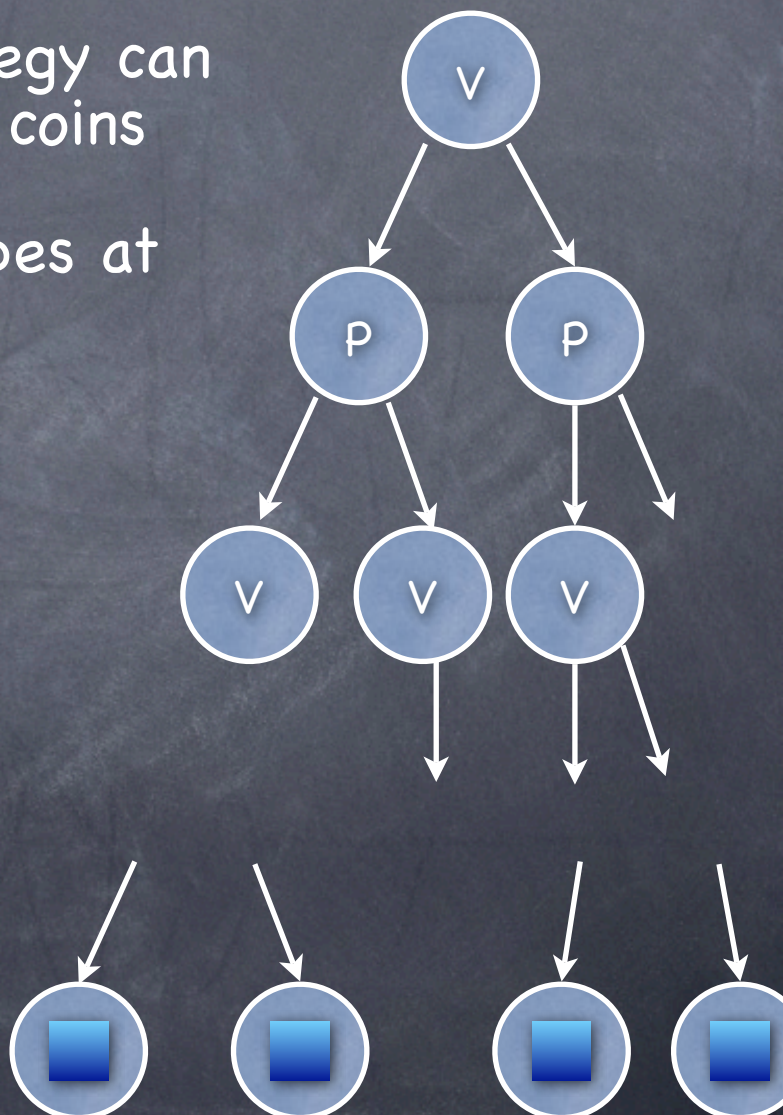

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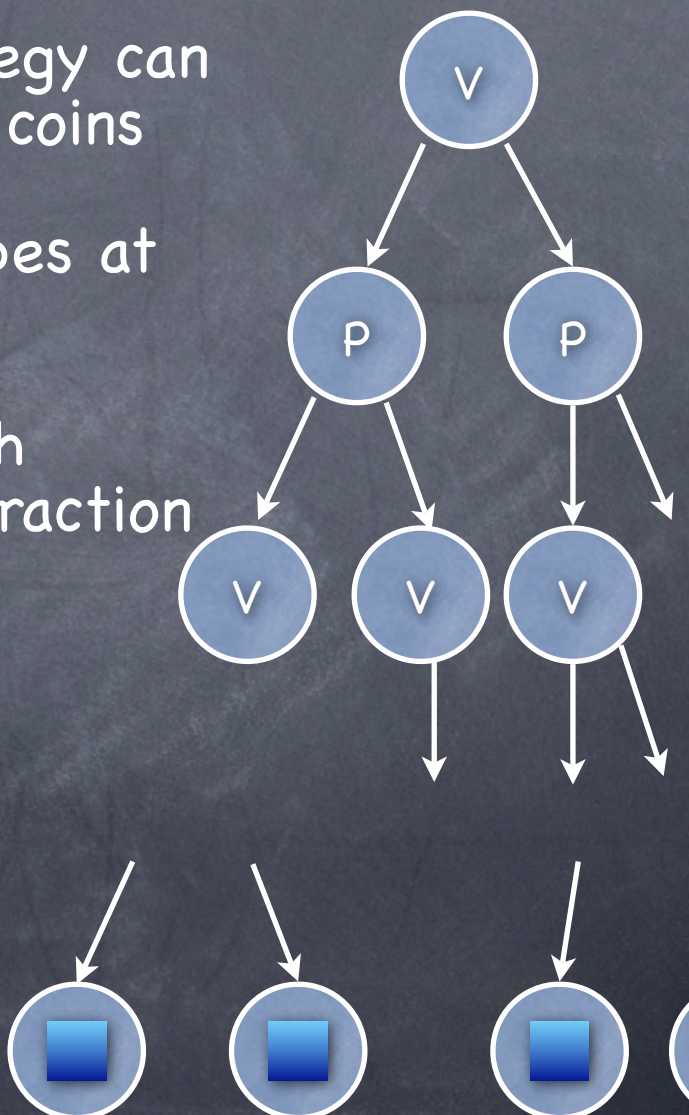

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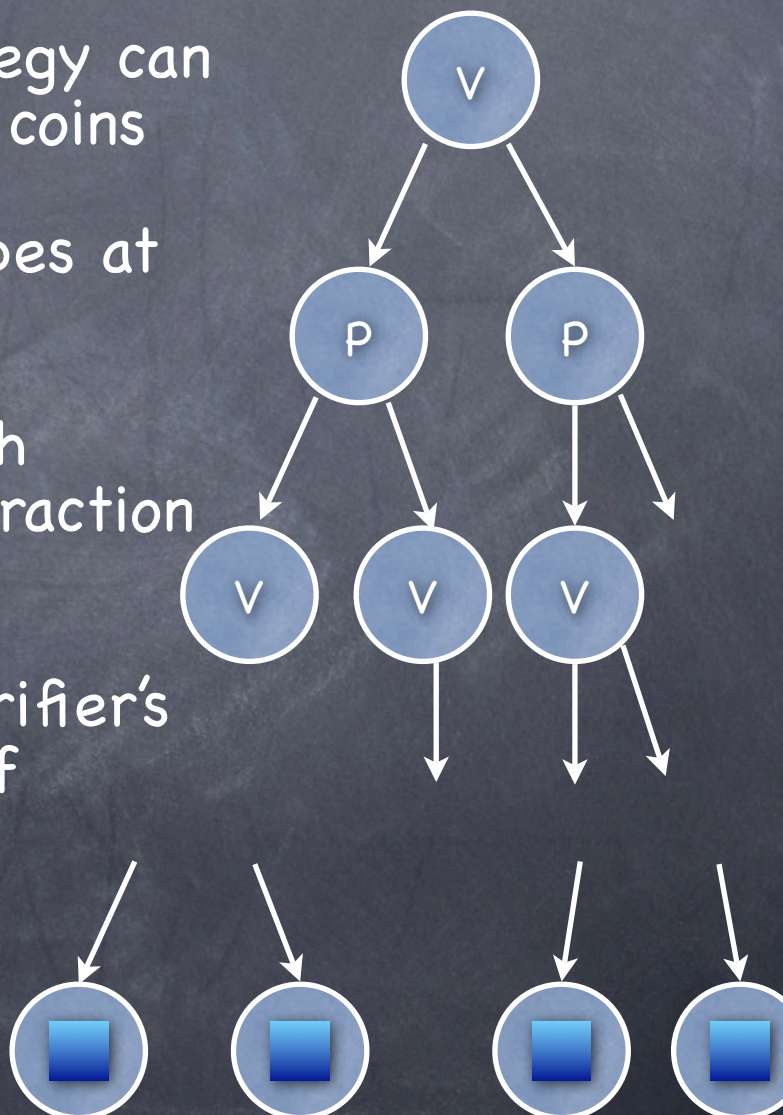

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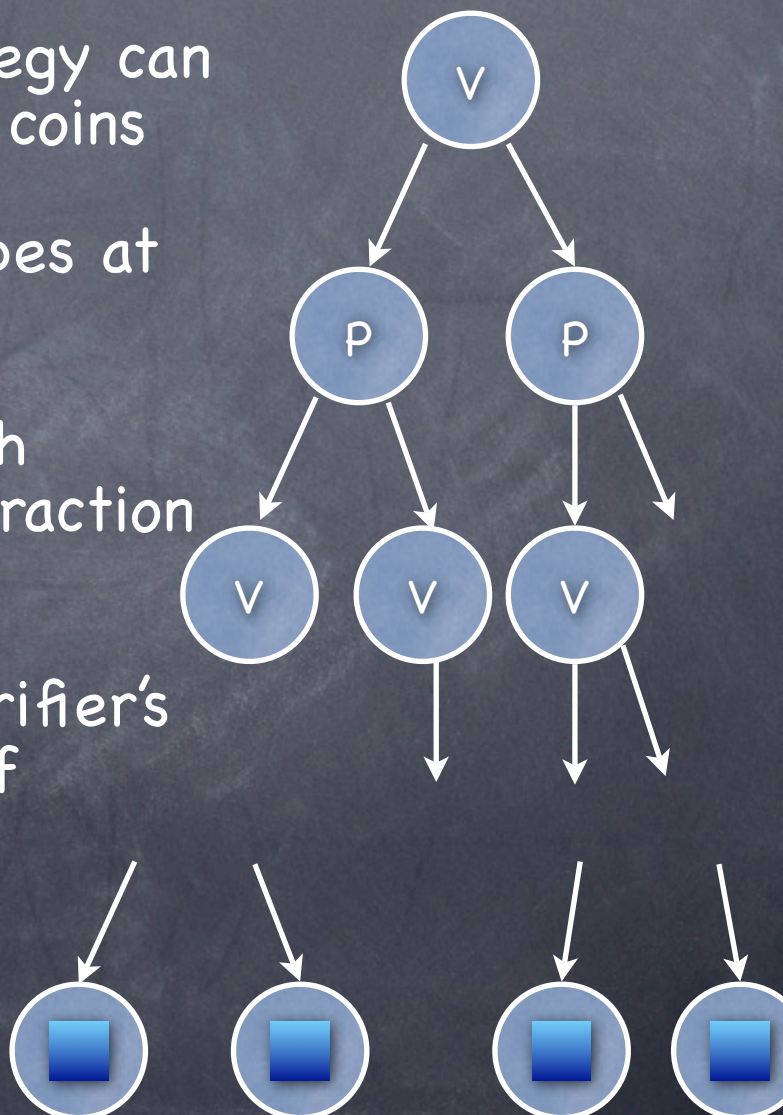

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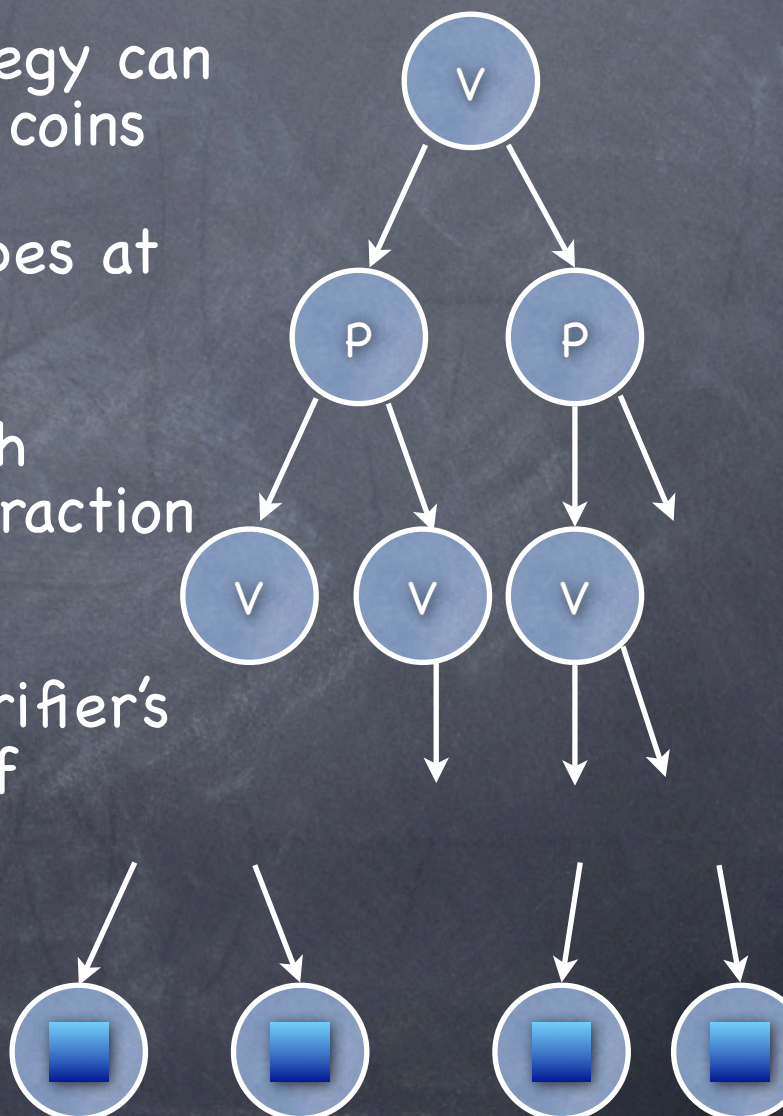

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  - QBF: $Q_1x_1\ Q_2x_2\ \ldots\ Q_nx_n\ F(x_1,\ldots,x_n)$ for quantifiers Q_i and a formula F on boolean variables

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      - Can always use a polynomial linear in each variable since $x^n=x$ for x=0 and x=1

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  - Consider proving = K (will be useful in the general case)

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  - Prover sends T=R (as d+1 coefficients) to verifier

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    - $\Sigma_{x1}...\Sigma_{xn} P(x_1,...,x_n) = R(0) + R(1)$
    - R has only one variable and degree at most d
  - Prover sends T=R (as d+1 coefficients) to verifier

Only Σ, no Π

Needs degree to be small

Sum-check protocol

Verifier has only oracle access to P

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  - Note: to evaluate need to add up 2^n values
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    - Note: P_1 has degree at most d; verifier has oracle access to P_1 (as it knows a, and has oracle access to P)

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        - At most nd/p if n variables. Can take p exponential.

IP Protocol for TQBF

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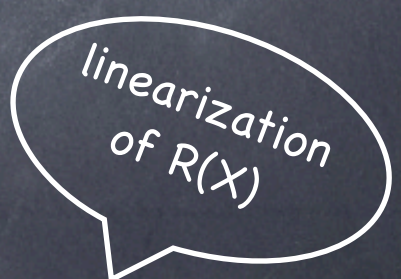

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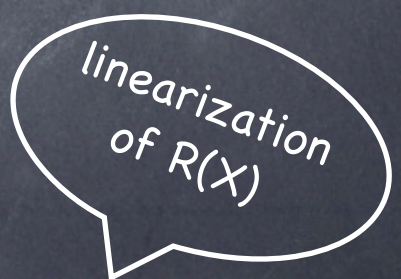

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