Probabilistic Computation

Lecture 15

Computing with Less Randomness, or with Imperfect Randomness

Soundness Amplification for BPP

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      - t = $O(n^d/\delta^2)$ enough for Pr[error] ≤ 2^{-n^d}

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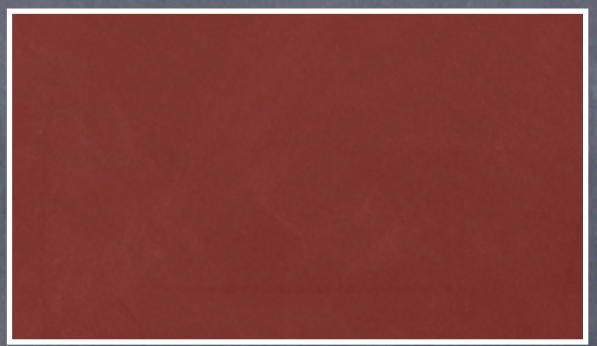

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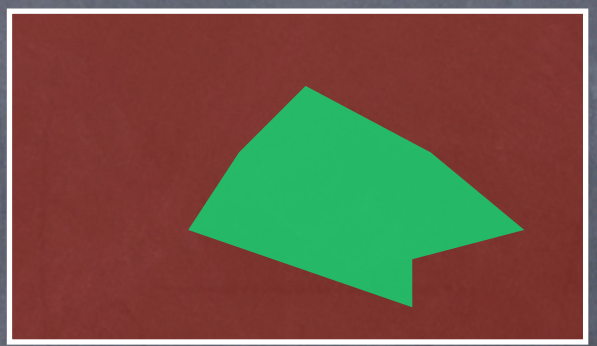

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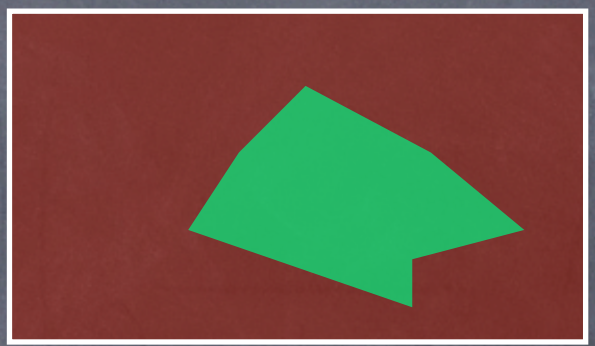

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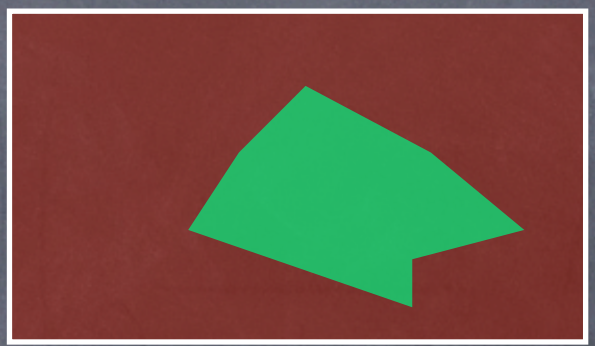

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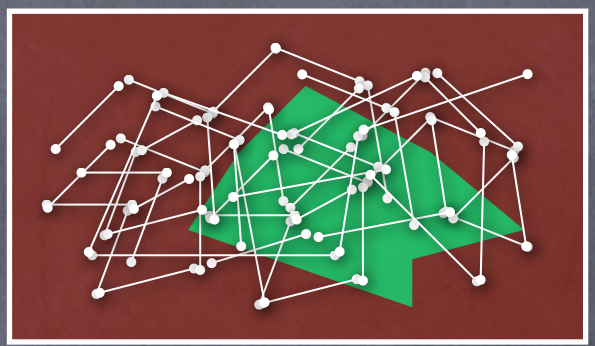

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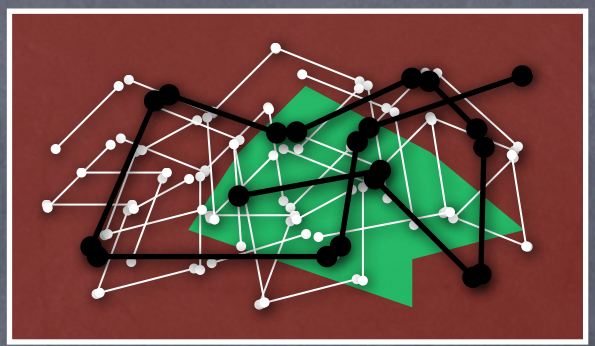

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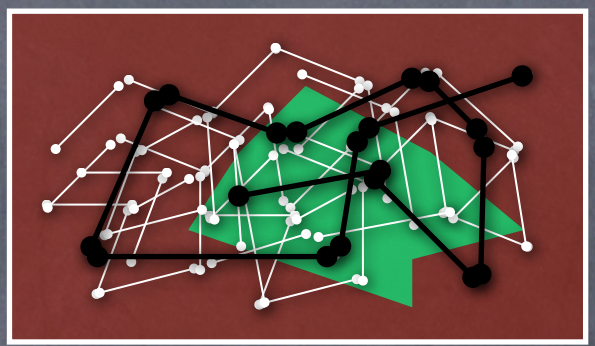

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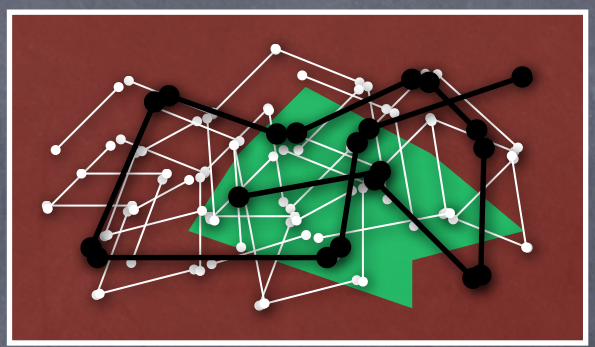

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    - Can we work with imperfect random sources?

Philosophical Issues with Randomness/Probability

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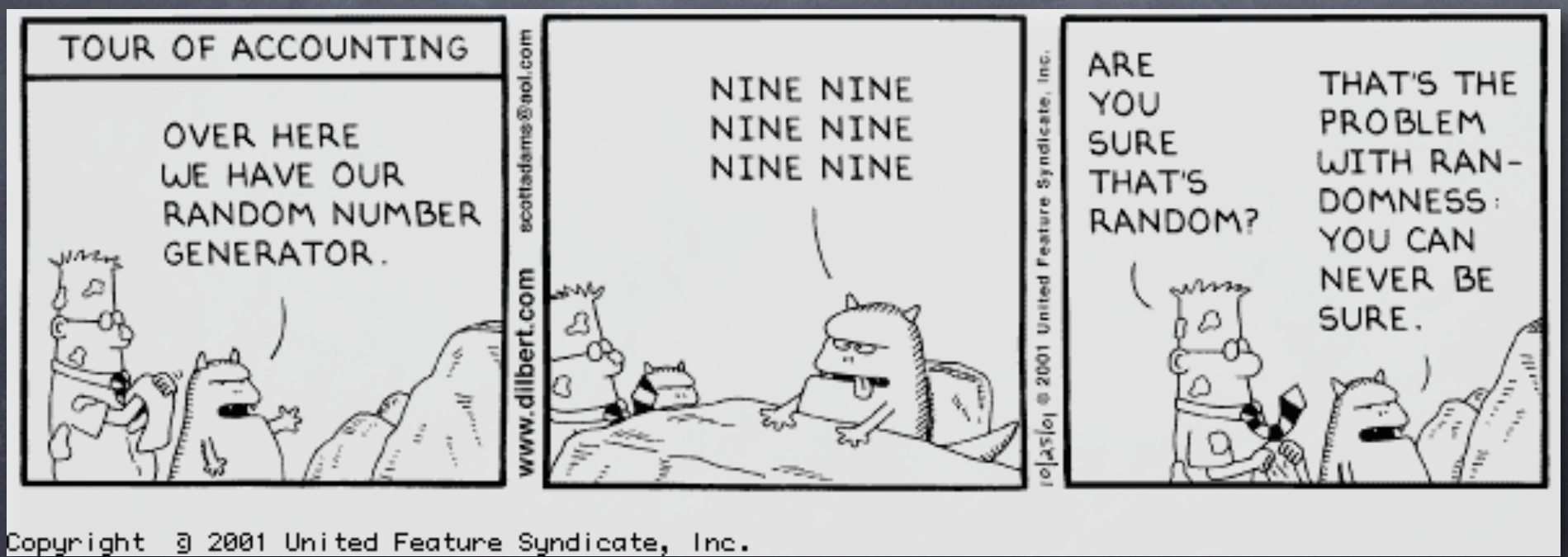

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- Weaker guarantees: e.g. Block source

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  - If on perfect randomness, Pr[error] < $1/(e2^n)$, then on imperfect randomness with bias < 1/m, Pr[error] < $1/2^n$

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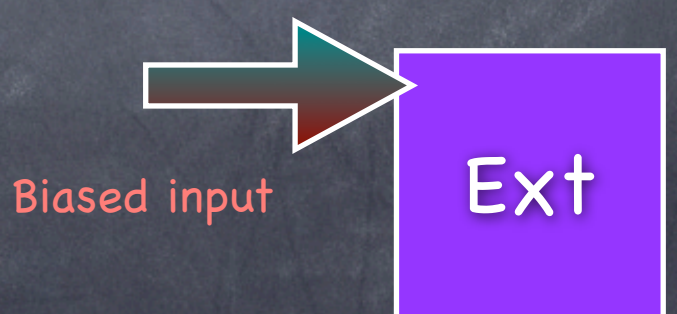

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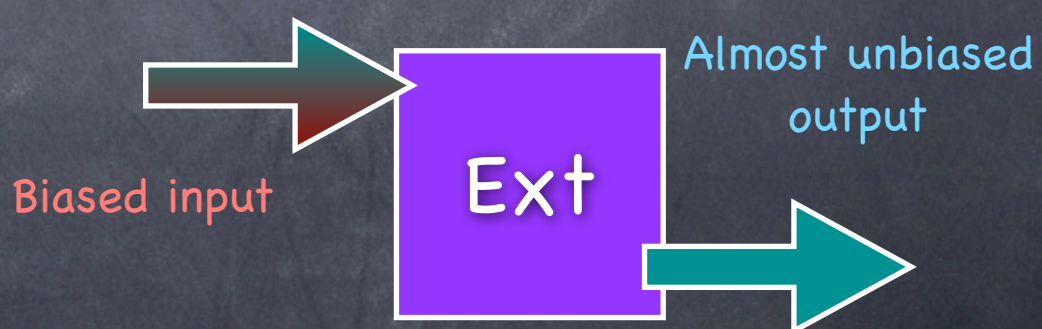

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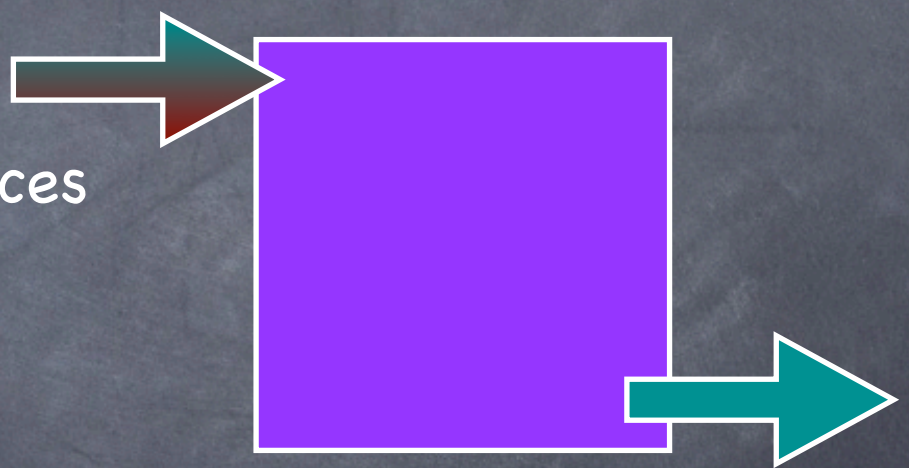

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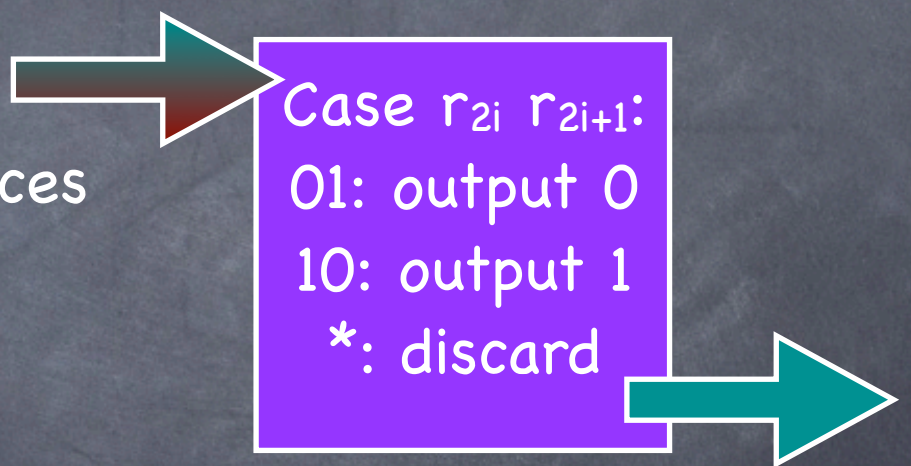

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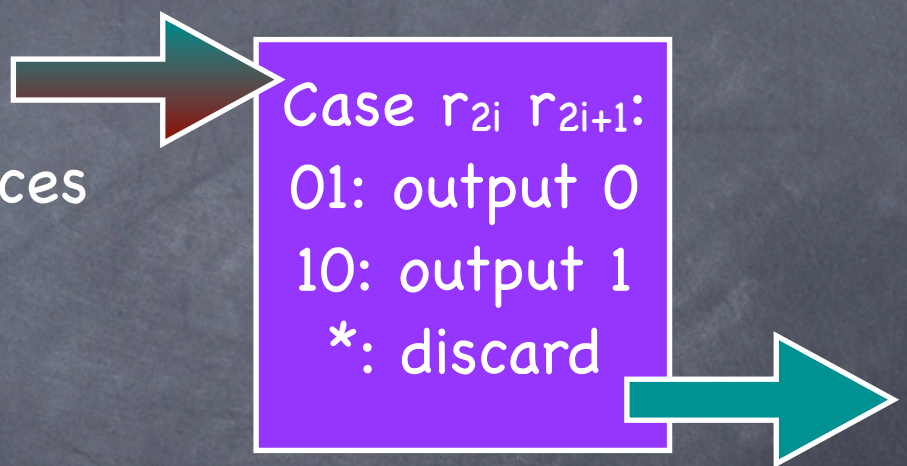

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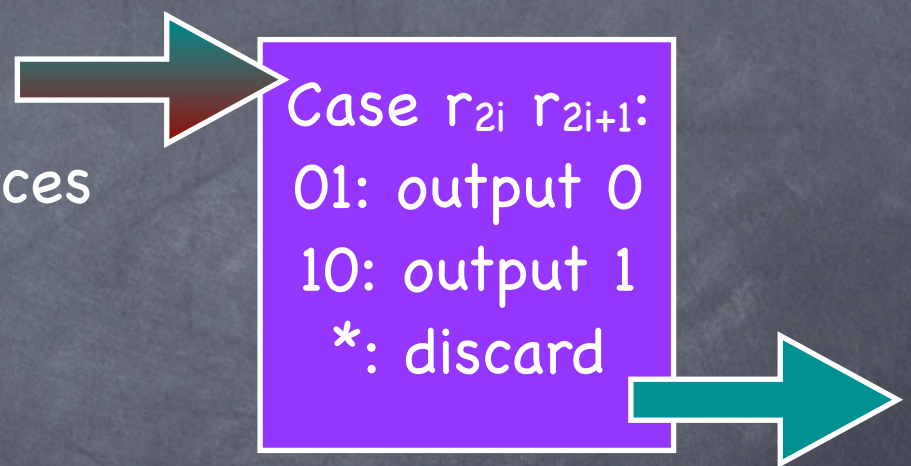

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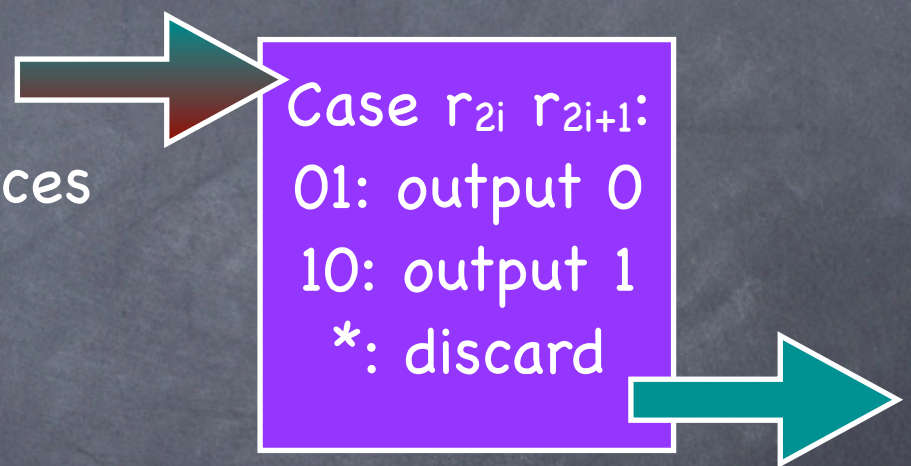

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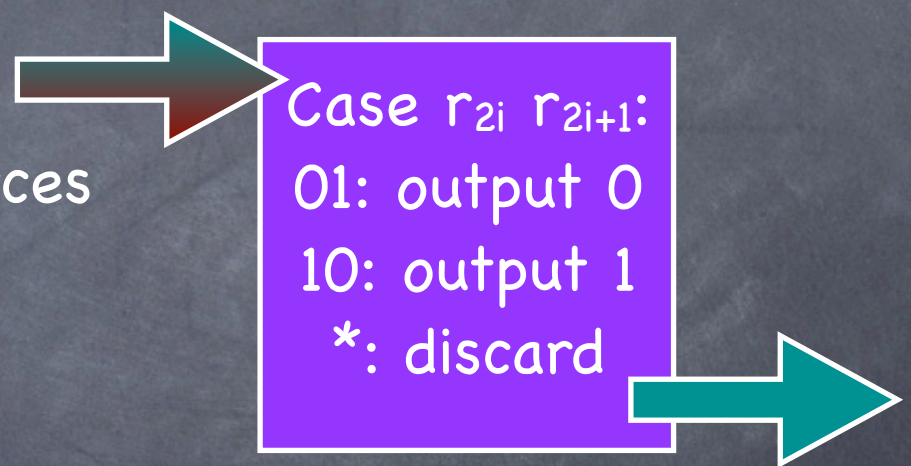

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  - Running time (per bit): constant number of tries, expected
- Can be generalized to sources which are (hidden) Markov chains

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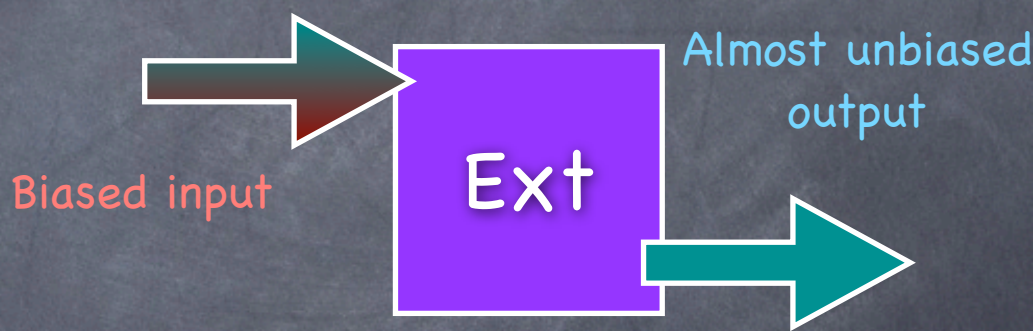

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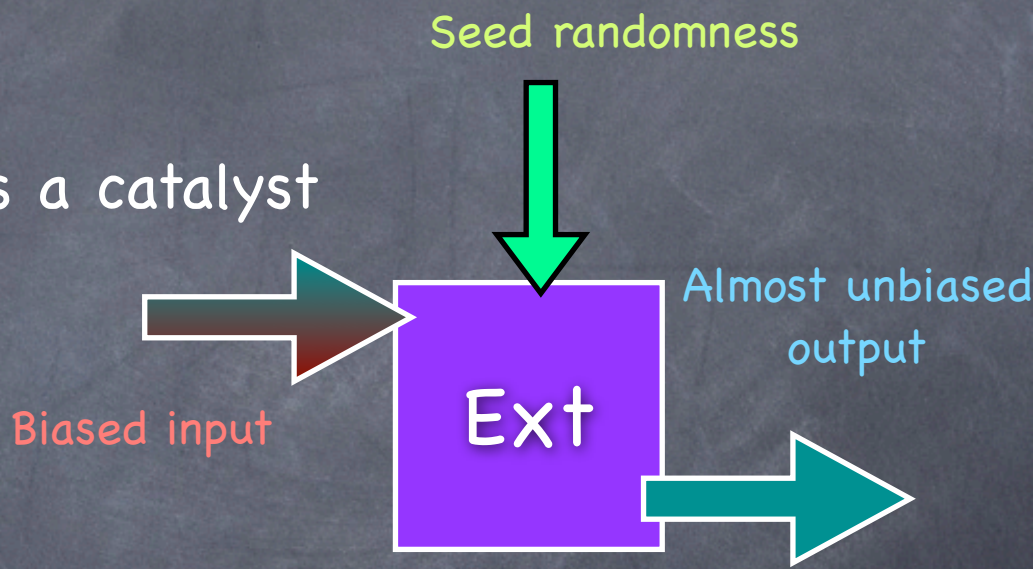

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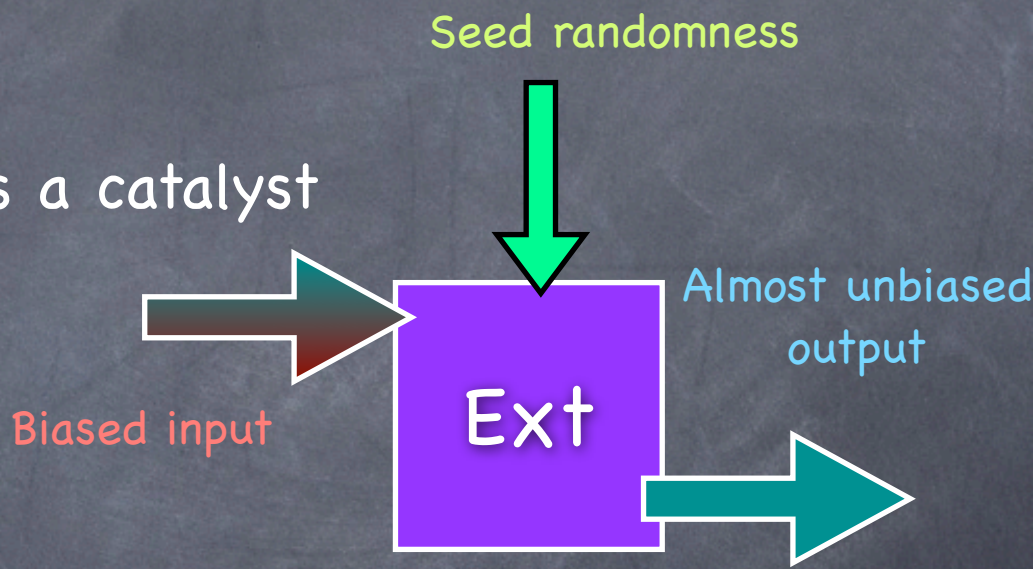

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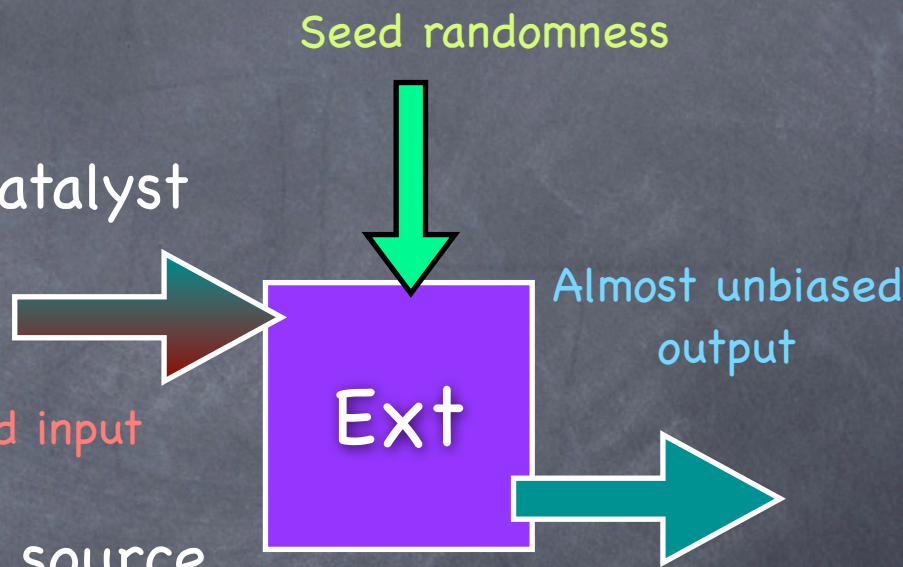
  - Draw one string from the biased source and generate random tapes, one for each seed. If the algorithm accepts on more than half the random tapes, accept.



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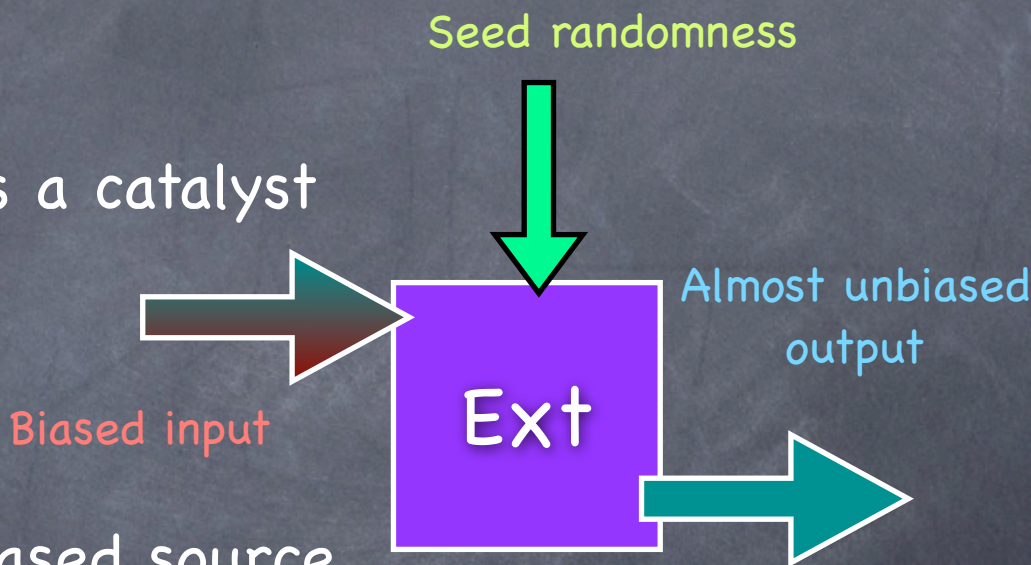

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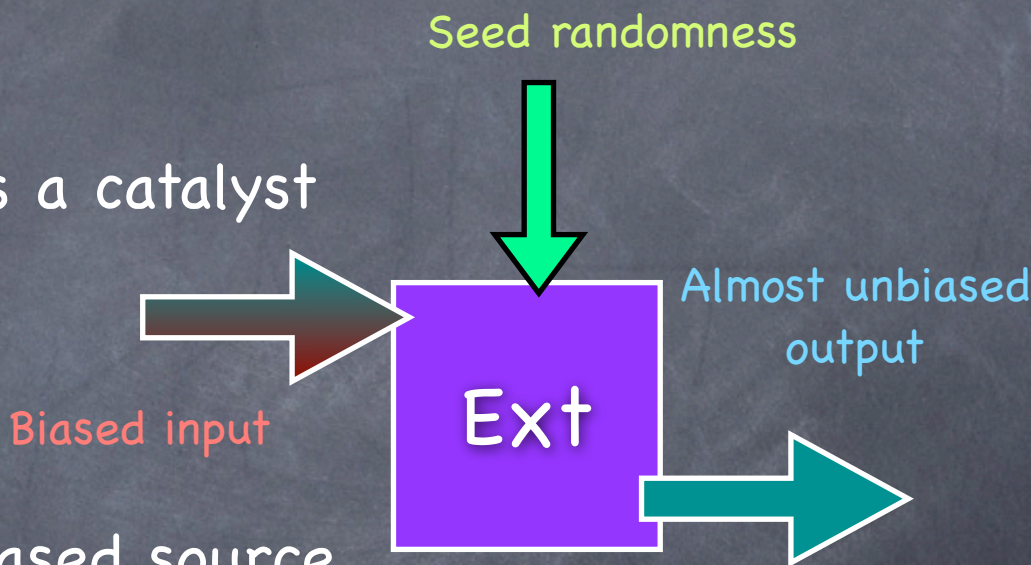

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    - Error probability remains bounded [Exercise]

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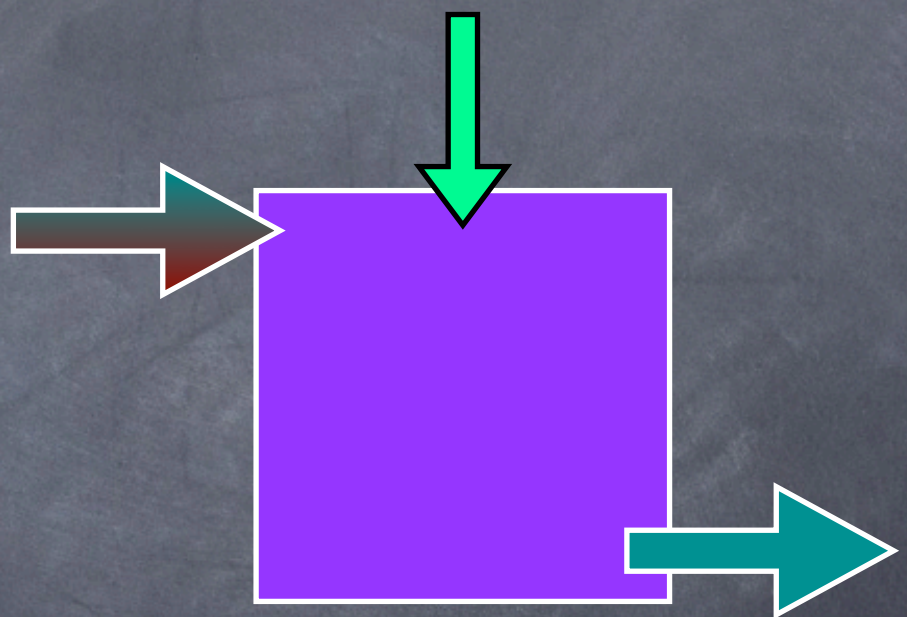

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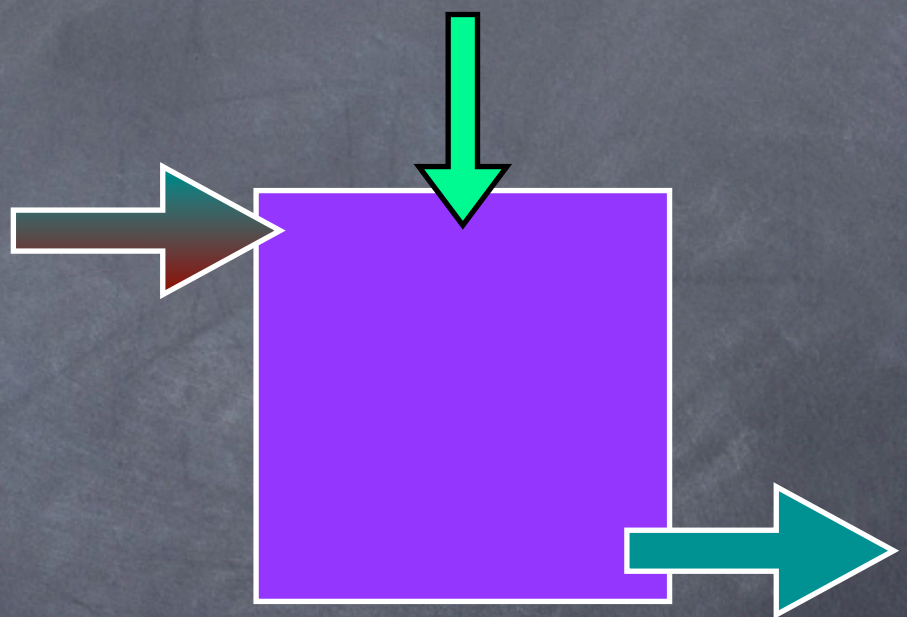

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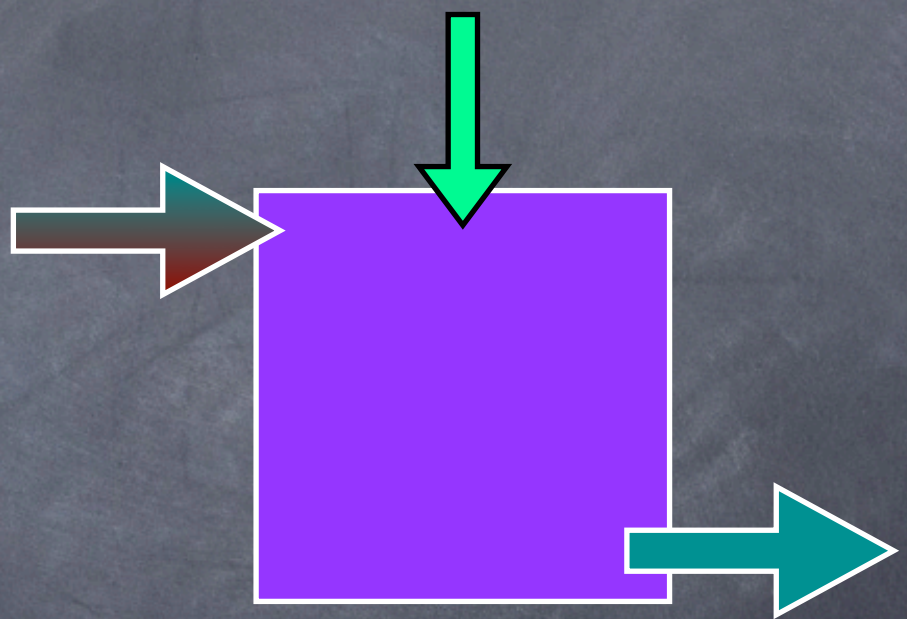
  - Input: SV(δ) for a constant δ<1



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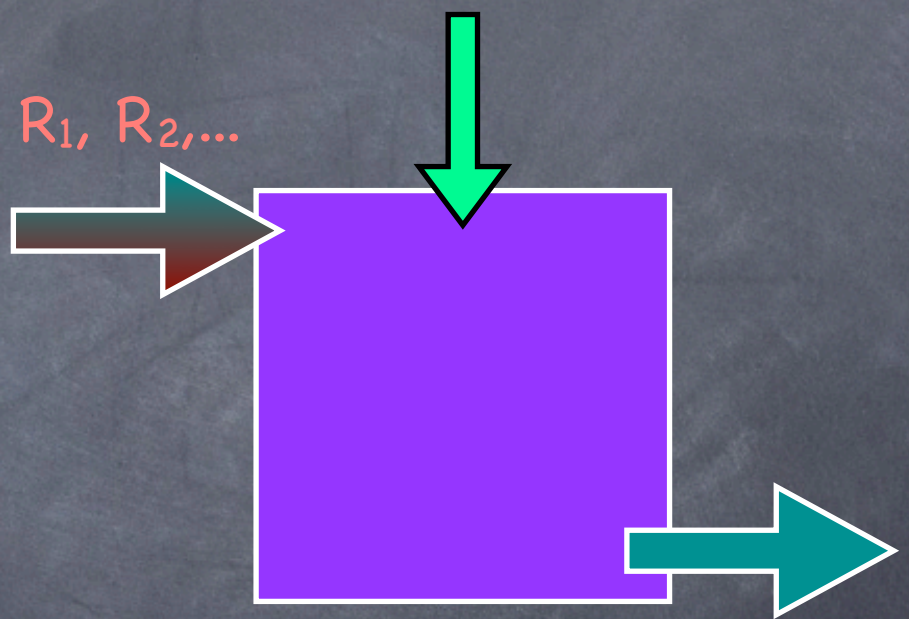

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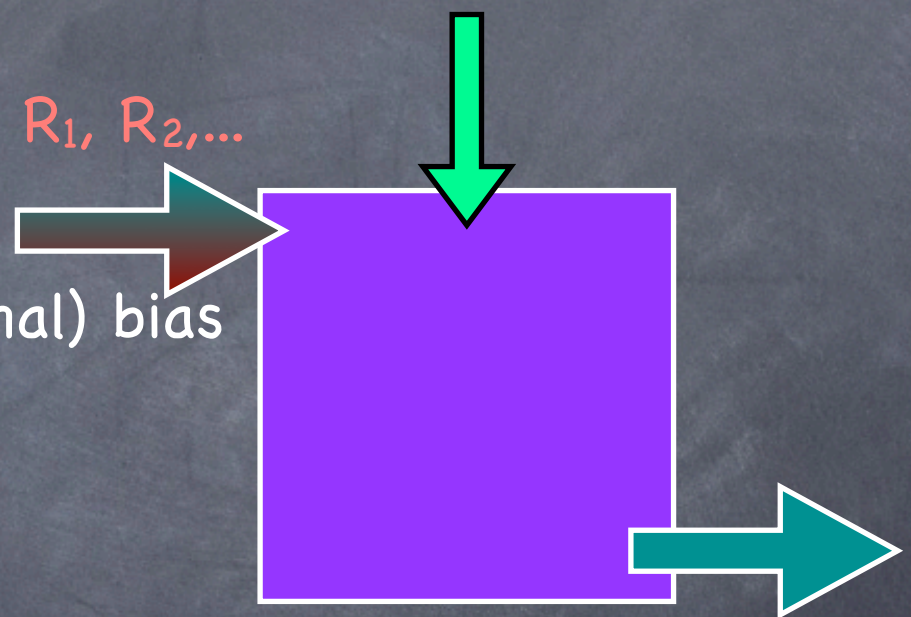

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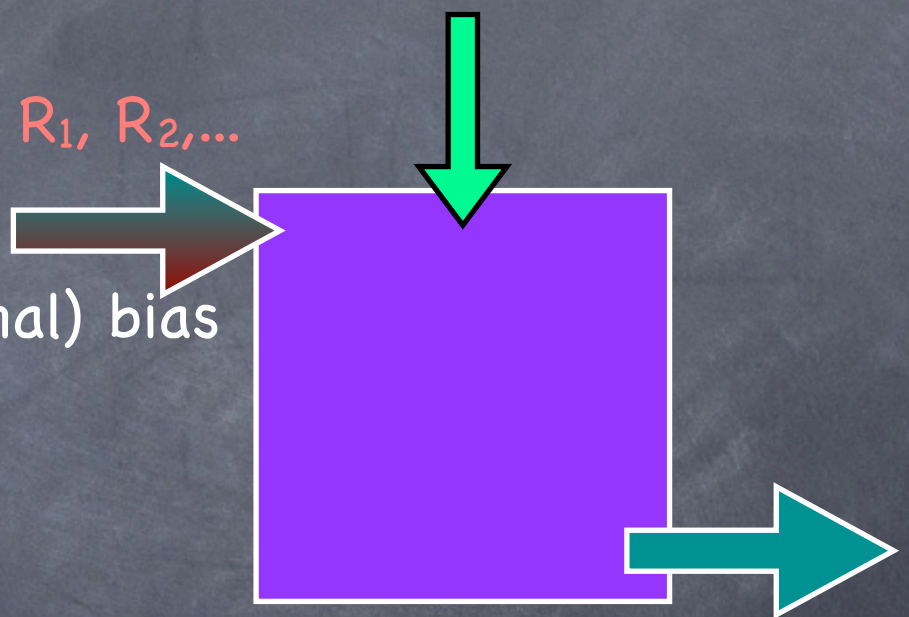

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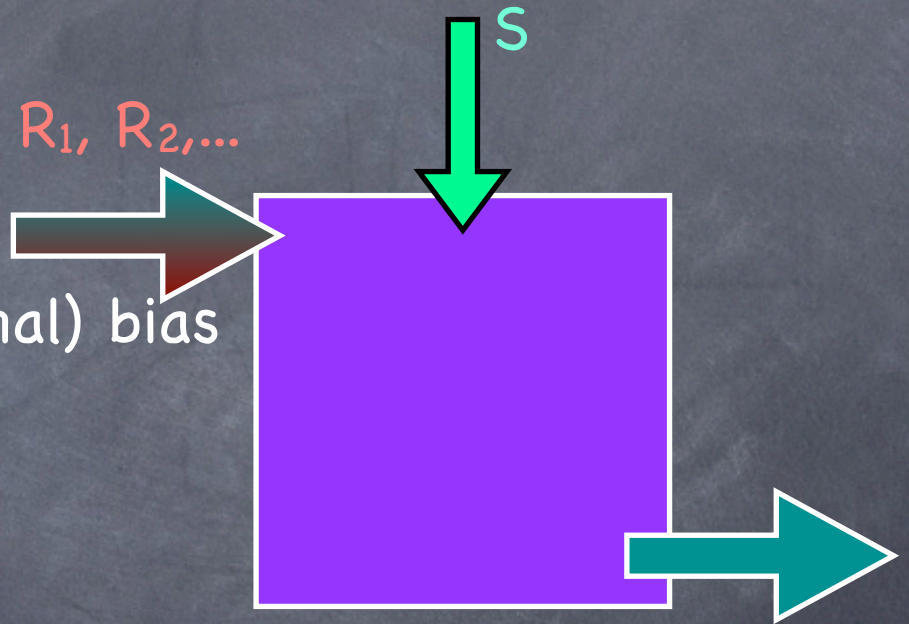

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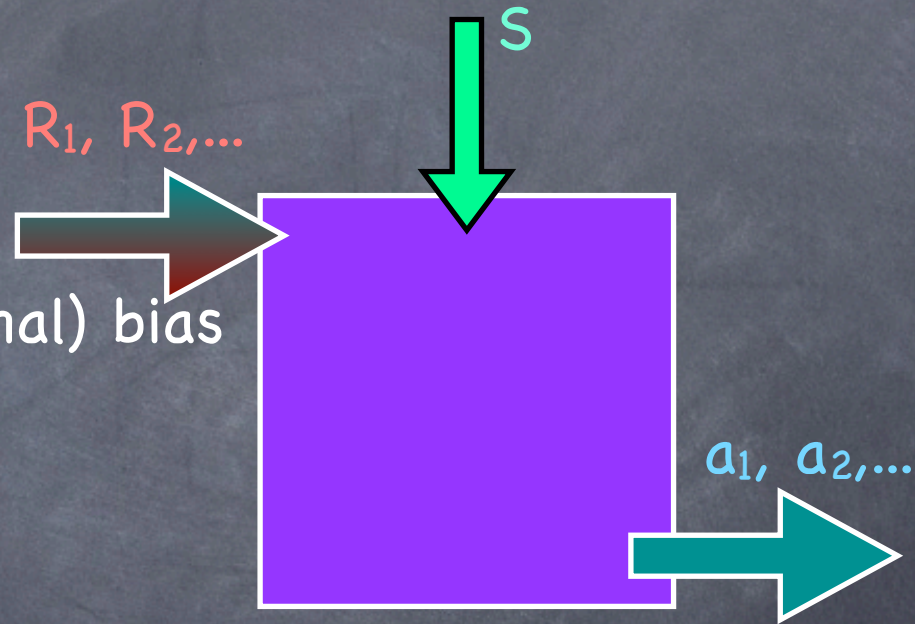

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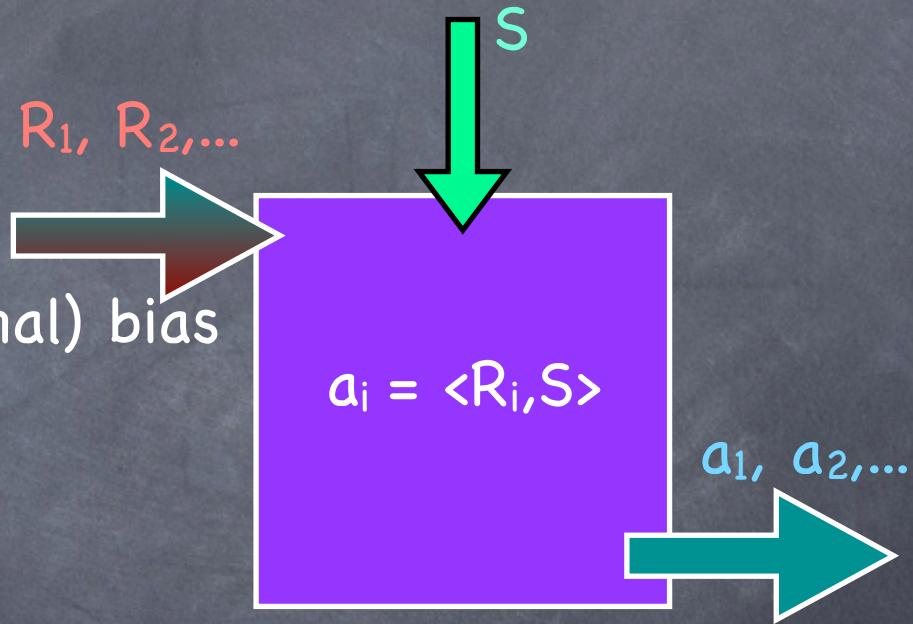

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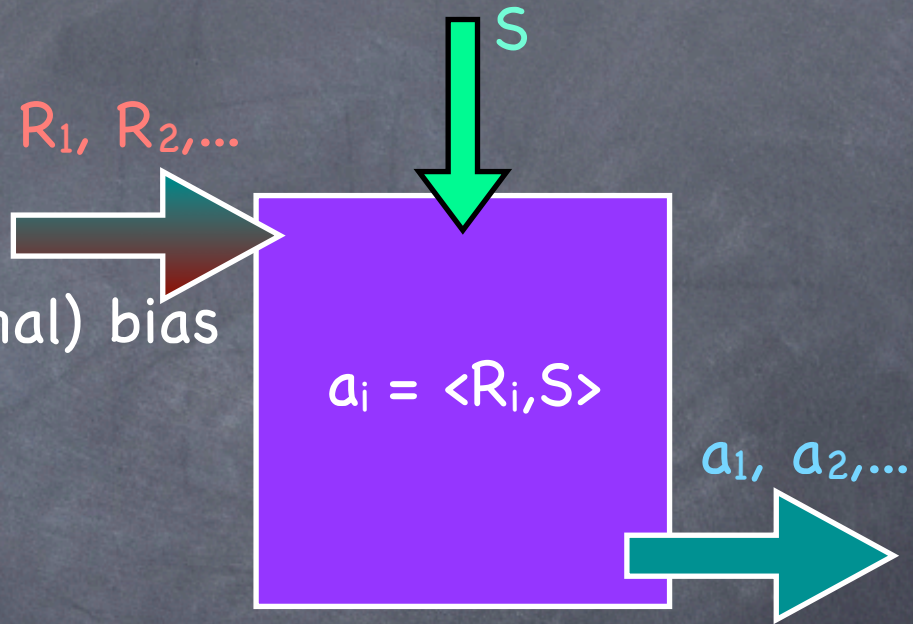

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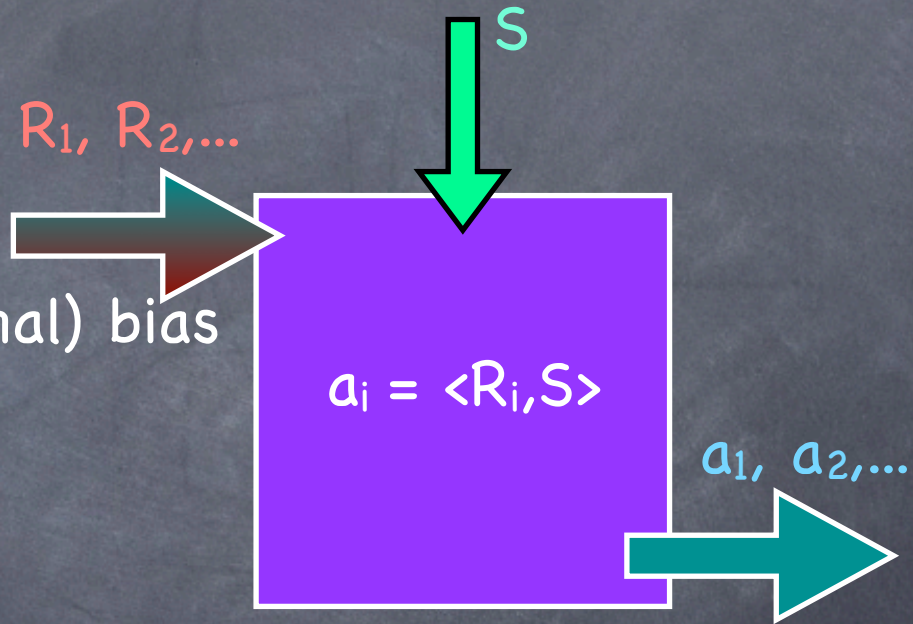

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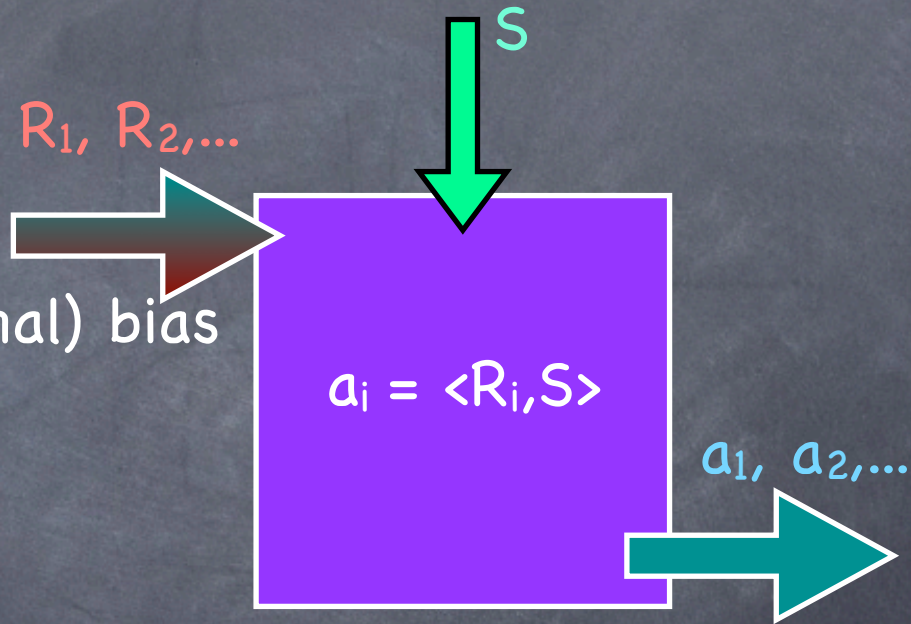

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- Bottom line: Can efficiently run BPP algorithms using very general classes of sources of randomness

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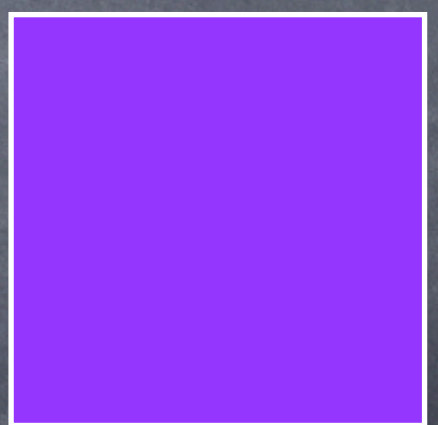

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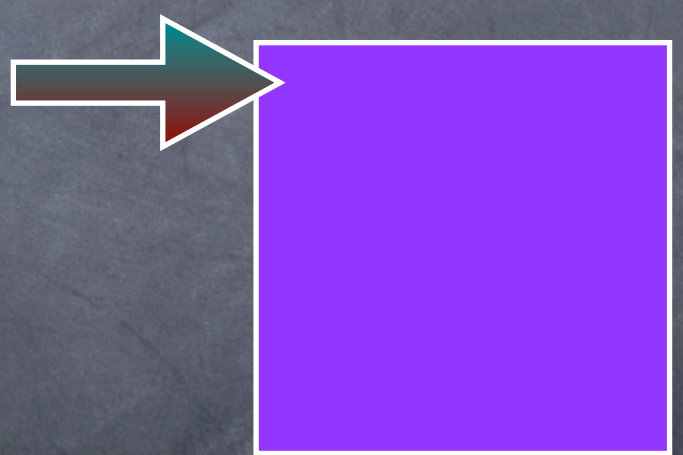

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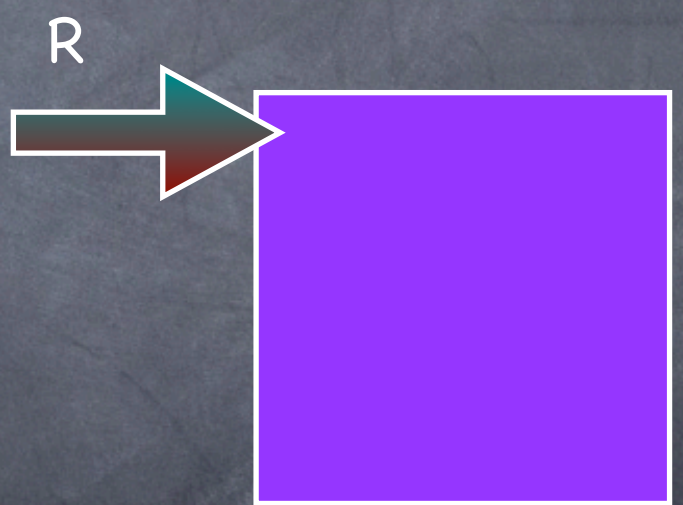

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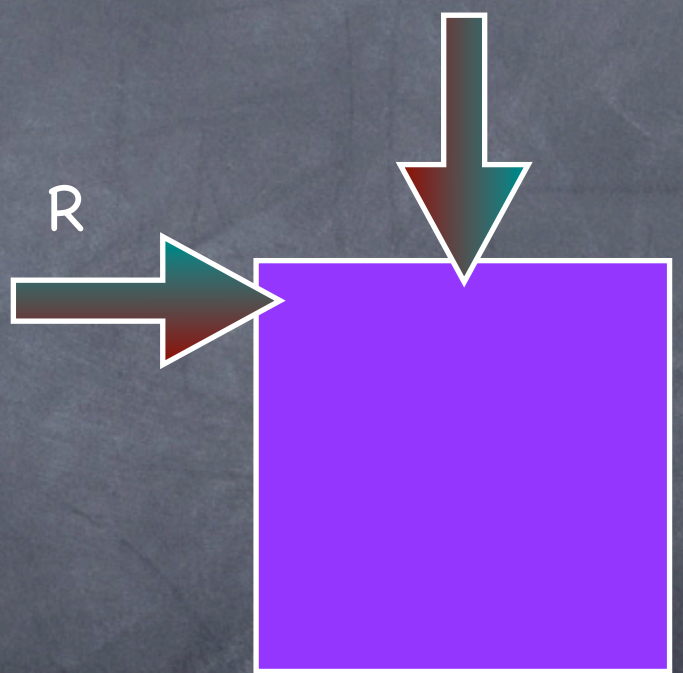

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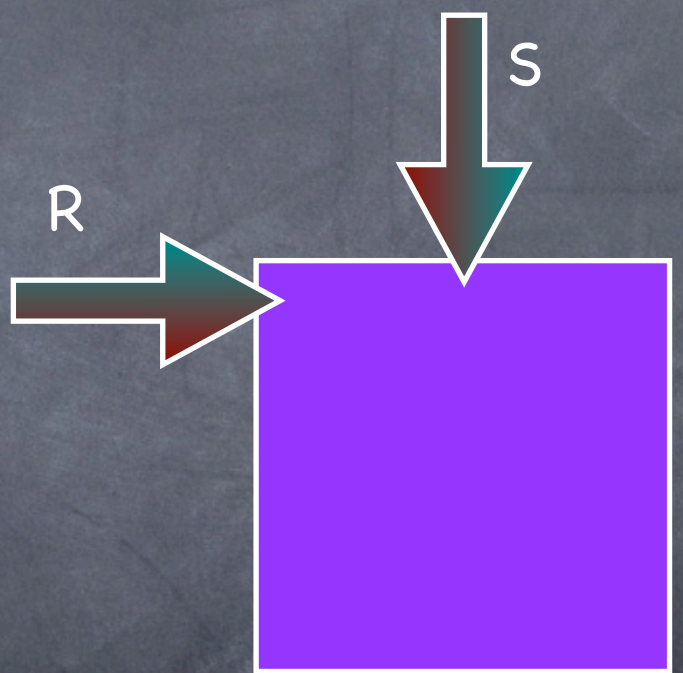

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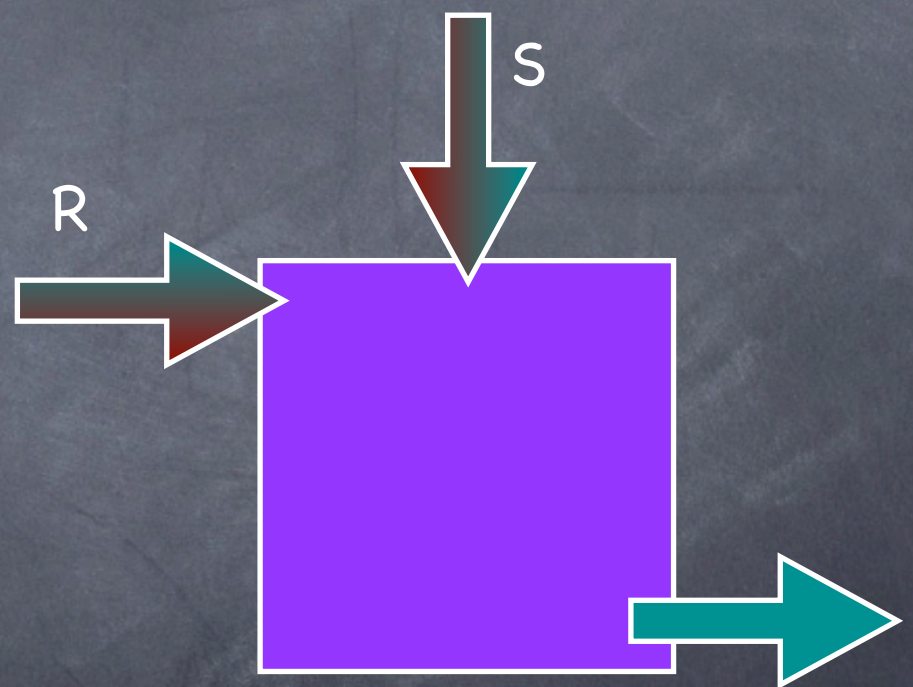

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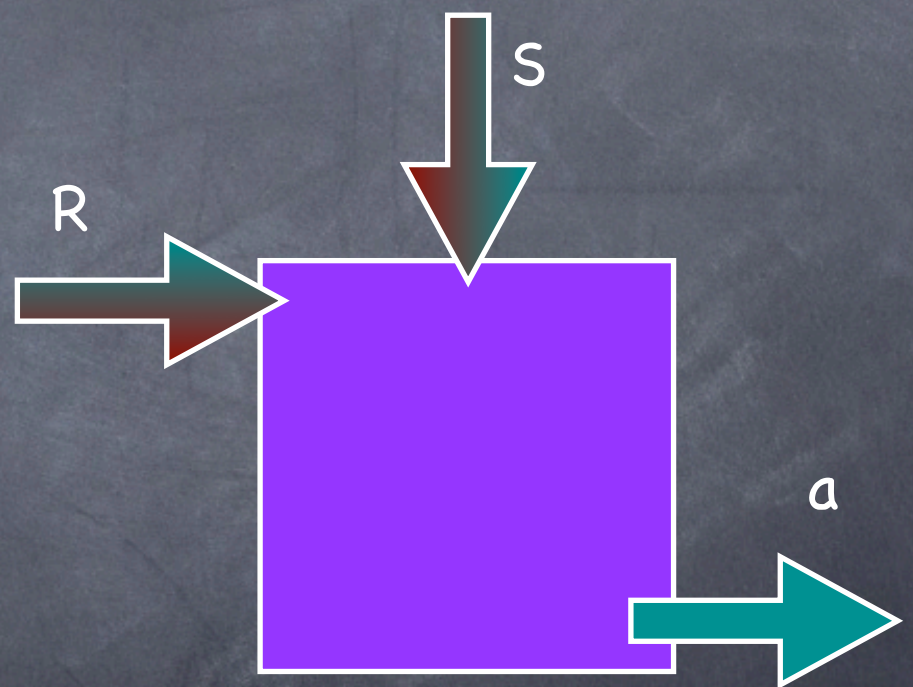

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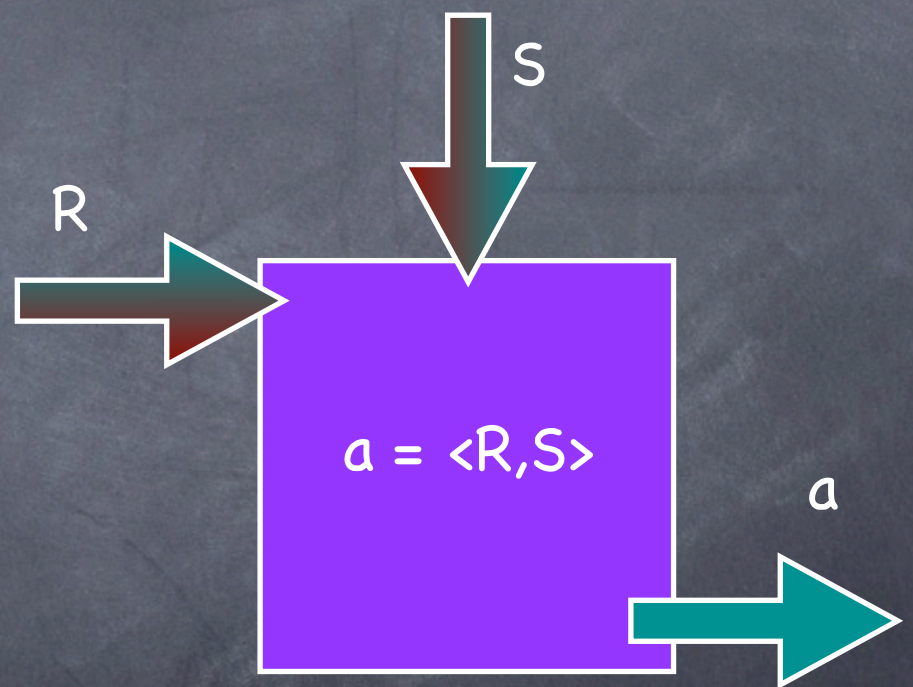

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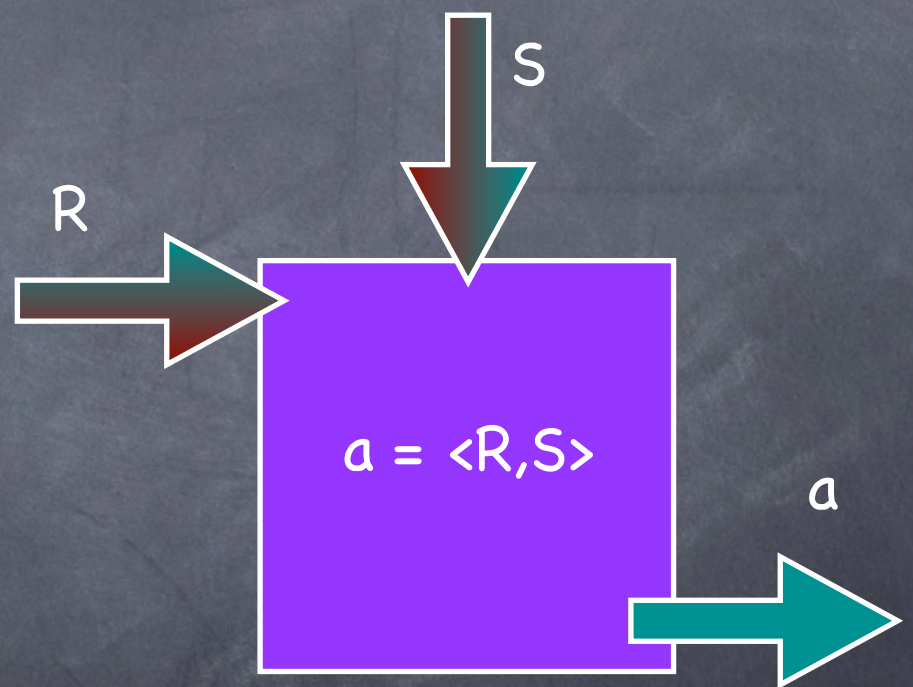

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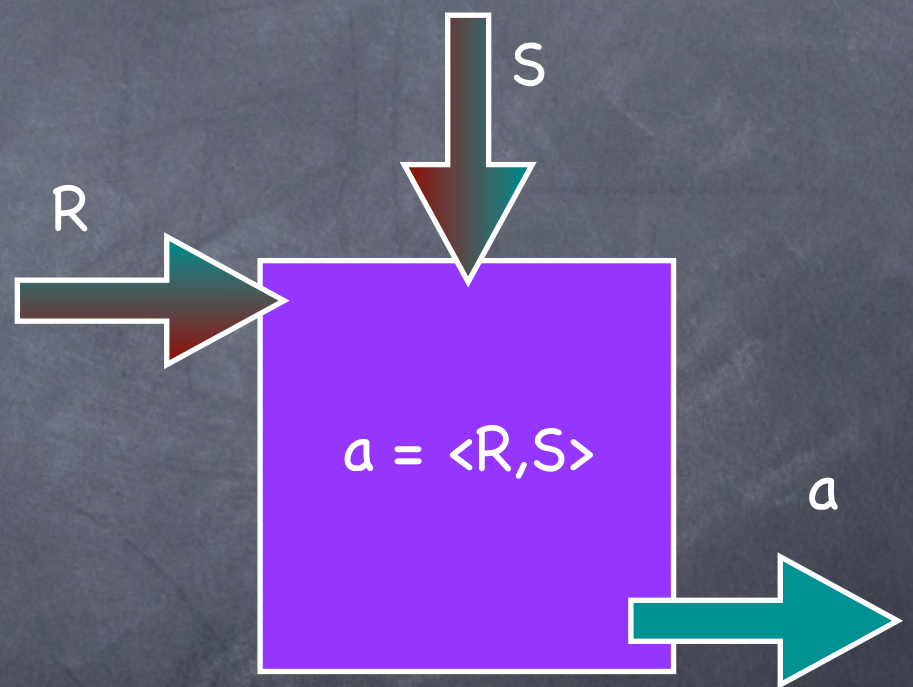

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