Complexity Homework 2

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Due: February 24, 2009

Problem 1:

We say that a complexity class $\mathbf{X}$ is *closed downward under Karp reductions* if:

$$\text{for all languages } A, B\text{: } A \in \mathbf{X} \text{ and } B \leq_P A \implies B \in \mathbf{X}$$

Show that $\mathbf{E}$ and $\mathbf{NE}$ are *not* closed downward under Karp reductions. (These two complexity classes were defined in the previous problem set.)

Problem 2:

A language A is *polynomial-time downward self-reducible* if there is a polynomial-time oracle machine M such that:

- $L(M^A) = A$. That is, when given an oracle for A, M decides A (self-reducibility).
- On input x, M only queries the oracle on strings *smaller* than x (downward reducibility).

The second restriction is necessary to make the property interesting – otherwise, on input x, M could just directly ask the oracle if $x \in A$.

(a) Show that SAT and TQBF are polynomial-time downward self-reducible.

(b) Show that if L is polynomial-time downward self-reducible, then $L \in \mathbf{PSPACE}$.

Problem 3:

Show that $\mathbf{P} \neq \mathbf{DSPACE}(n)$.

Hint: Show that one class is closed downward under Karp reductions, while the other is not.

Problem 4:

An oracle machine is called a *robust oracle machine* if the language accepted by it remains the same no matter which oracle is used (however the running time may vary). Show that a language L is decided by M^K *in polynomial time* where M is a robust oracle machine and K is some oracle, if and only if $L \in \mathbf{NP} \cap \mathbf{co\text{-}NP}$.

Problem 5:

Consider any game between two parties Alice and Bob which will terminate in a finite number of steps, with only two possible outcomes, say 0 and 1.

In the class we saw that any such game falls into one of the two following kinds: (a) games in which Alice has a strategy to force the outcome to be 0, and (b) games in which Bob has a strategy to force the outcome to be 1. (The strategies may be hard to compute, given the description of the game.)

In this problem you will prove a more general result.

1. Show that any game falls into one of the four categories:
   (a) Games in which Alice has a strategy to force the outcome to be 0 *and* she has a strategy to force the outcome to be 1.
   (b) Games in which Bob has a strategy to force the outcome to be 0 *and* he has a strategy to force the outcome to be 1.
   (c) Games in which both Alice and Bob have strategies to force the outcome to be 0.
   (c) Games in which both Alice and Bob have strategies to force the outcome to be 1.

   Hint: Use induction on the number of steps in the protocol.
2. Use this to conclude that in any game in which either Alice or Bob "wins," either Alice has a winning strategy or Bob has a winning strategy.