CS 579: Computational Complexity. Lecture 11

Expansion and Eigenvalues

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In the next few minutes:

Why spectral graph theory is both natural and magical
Representing Graphs

Obviously, we can represent a graph with an nxn matrix

Adjacency matrix

\[ A_{ij} = \begin{cases} 
  w_{ij} & \text{weight of edge } (i, j) \\
  0 & \text{if no edge between } i, j 
\end{cases} \]
Representing Graphs

V: n nodes
E: m edges
G = {V,E}

What is not so obvious, is that once we have matrix representation view graph as **linear operator**

- Can be used to multiply vectors.
- Vectors that don’t rotate but just scale = eigenvectors.
- Scaling factor= eigenvalue

Amazing how this point of view gives information about graph

\[ A : \mathbb{R}^n \rightarrow \mathbb{R}^n \]

\[ Ax = \mu x \]
“Listen” to the Graph

List of eigenvalues
\{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n\}:

adjacency matrix

\[ A = \begin{bmatrix}
wij \\
\end{bmatrix}\]

Eigenvalues reveal **global** graph properties not apparent from edge structure

A drum:

**Hear** shape of the drum
“Listen” to the Graph

List of eigenvalues
{μ₁ ≥ μ₂ ≥ ... ≥ μₙ}

Eigenvalues reveal **global** graph properties not apparent from edge structure

Adjacency matrix

\[
A = \begin{pmatrix}
   & & & & \\
   & & & & \\
   & & & & \\
   & & & & \\
\end{pmatrix}
\]

Hear shape of the drum
Its sound:
“Listen” to the Graph

List of eigenvalues
\{ \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \}: graph SPECTRUM

Eigenvalues reveal **global** graph properties
not apparent from edge structure

Hear shape of the drum

Its sound
(eigenfrequencies):
“Listen” to the Graph

List of eigenvalues
{\( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \)}: graph SPECTRUM

Eigenvalues reveal global graph properties not apparent from edge structure

If graph was a drum, spectrum would be its sound
Eigenvectors are Functions on Graph

\[ v \in \mathbb{R}^n, \quad v : V \rightarrow \mathbb{R} \]

\[ Av = \mu v \]

\[ v(i) = \text{value at node } i \]
Eigenvectors are Functions on Graph

\[ v \in \mathbb{R}^n, \quad v : V \rightarrow \mathbb{R} \quad \text{Av} = \mu v \]

\[ v(i) = \text{value at node } i \quad \text{different shade of grey} \]
So, let’s See the Eigenvectors
The second eigenvector
Third Eigenvector
Fourth Eigenvector

* Slides from Dan Spielman
Cuts and Algebraic Connectivity

Cuts in a graph:

\[
cut(S, S') = \frac{E(S, S')}{|S|}, |S| \leq n/2
\]

Graph not well-connected when “easily” cut in two pieces
Cuts and eigenvalues

Edge-expansion:

$$h(G) = \min_{S:|S| \leq n/2} \frac{E(S, \bar{S})}{|S|}$$

Graph not well-connected when “easily” cut in two pieces

Would like to know Sparsest Cut but NP hard to find

How does algebraic connectivity relate to standard connectivity?

**Theorem** (Cheeger-Alon-Milman): \( \lambda_2 \leq h(G) \leq \sqrt{2d_{\text{max}}} \sqrt{\lambda_2} \)
Today

- More on evectors and evals.
- Evalshes of d-regular graphs.
- Relation between eigenvalues and expansion (Cheeger, part 1).
A Remark on Notation

For convenience, we will often use the bra-ket notation for vectors:

- We denote vector $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ with a “bra”: $|\mathbf{v}\rangle$

- We denote the transpose vector $\mathbf{v}^T = (v_1 \ldots v_n)$ with a “ket”: $\langle \mathbf{v} |$

- We denote the inner product $\mathbf{v}^T \mathbf{u}$ between two vectors $\mathbf{v}$ and $\mathbf{u}$ with a “braket”: $\langle \mathbf{v} | \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
E vectors and E values

- Vector $v$ is an eigenvector of matrix $M$ with eigenvalue $\lambda$ if $Mv = \lambda v$.
- We are interested (almost always) in symmetric matrices, for which the following special properties hold:
  - If $v_1, v_2$ are eigenvectors of $A$ with eigenvalues $\lambda_1, \lambda_2$ and $\lambda_1 \neq \lambda_2$, then $v_1$ is orthogonal to $v_2$. (Proof)
  - If $v_1, v_2$ are eigenvectors of $A$ with the same eigenvalue $\lambda$, then $v_1 + v_2$ is as well. The multiplicity of eigenvalue $\lambda$ is the dimension of the space of eigenvectors with eigenvalue $\lambda$.
  - Can assume that eigenvectors have unit length, since every multiple of an eigenvector is also an eigenvector.
E_vectors and E_values

- Generally,\n  \[ Mv = \lambda v \Rightarrow (M - \lambda I)v = 0 \Rightarrow \det(M - \lambda I) = 0. \]
- The determinant is an n-degree polynomial and has n roots, counting multiplicities.
- Every n-by-n symmetric matrix has n evvalues \( \{\lambda_1 \leq \cdots \leq \lambda_n\} \) counting multiplicities, and and orthonormal basis of corresponding e_vectors \( \{v_1, \ldots, v_n\} \), so that \( Mv_i = \lambda_i v_i \)

- If we let V be the matrix whose i-th column is \( v_i \), and D the diagonal matrix whose i-th diagonal is \( \lambda_i \), we can compactly write \( MV = VD \). Multiplying by \( V^T \) on the right, we obtain the eigendecomposition of M:
  \[ M = MV V^T = VD V^T = \sum_i \lambda_i v_i v_i^T \]
Some eigenvalue theorems

Theorem 1. Let $M \in R^{n \times n}$ symmetric. Then
$$\lambda_1 = \max_{x \in R^n, \|x\|=1} \{x^T M x\},$$
where
$$x^T M x = \sum_{i,j} x(i)x(j)M(i,j).$$

Similarly, $\lambda_2 = \max_{x \in R^n, \|x\|=1, x \perp x_1} \{x^T M x\}$

$$\max\{|\lambda_2|, \ldots, |\lambda_n|\} = \max_{x \in R^n, \|x\|=1} \{|x^T M x|\}.$$
Some eigenvalue theorems

**Theorem 2.** Let $G$ be a $d$-regular graph and $M$ its adjacency matrix. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues and $x_1, x_2, \ldots, x_n$ the corresponding eigenvectors. Then $\lambda_1 = d$. Moreover, $x_1 = (1, \ldots, 1)$. 
Eigenvalues and connectivity

- **Theorem 2′.** Let $G$ be a $d$-regular graph and $M$ its adjacency matrix. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues and $x_1, x_2, \ldots, x_n$ the corresponding eigenvectors. Then $\lambda_1 = d$. If $\lambda_2 = d$ then the graph is disconnected. The converse is also true (ex). Alternatively, $h(G)= 0$ iff $\lambda_2 = d$.

- Generally, the more connected the graph is, the smaller $\lambda_2$ is.
Eigenvalues and expansion

- **Cheeger’s Inequality:**

\[
\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{d(d - \lambda_2)}
\]

- Both upper and lower bounds are tight (up to constant), as seen by path graph and complete binary tree.