Problem 1

(10 pts.) Show that \( \text{SPACE}(n) \neq \text{NP} \).

Problem 2

Recall that \( E = \text{DTIME}(2^{O(n)}) \) is the class of problems solvable by deterministic turing machine in time \( 2^{O(n)} \), where \( n \) is the length of the input. We say that a language \( A \) has a many-to-one polynomial time reduction to a language \( B \), written \( A \leq_{m^p} B \) if there is a polynomial time computable function \( f(\cdot) \) such that for every instance \( x \) we have \( x \in A \iff f(x) \in B \).

• (10 pts.) Show that \( \text{NP} \) is closed under polynomial many-to-one reductions, that is \( A \leq_{m^p} B \) and \( B \in \text{NP} \) implies \( A \in \text{NP} \).

• (10 pts.) Show that if \( E \) were closed under many-to-one reductions, we would have a contradiction to the time hierarchy theorem. Conclude that \( \text{NP} \neq E \).

Problem 3

Let \( S = \{S_1, \ldots, S_m\} \) be a collection of subsets of a finite set \( U \). The Vapnik-Chervonenkis (VC) dimension of \( S \), denoted \( VC(S) \), is the size of the largest set \( X \subseteq U \) such that for every \( X' \subseteq X \), there is an \( i \) for which \( S_i \cap X = X' \). A Boolean circuit succinctly represents the collection \( S \), if \( S_i \) consists of exactly those elements \( x \in U \) for which \( C(i, x) = 1 \).

Let \( \text{VC-DIMENSION} = \{\langle C, k \rangle : C \text{ represents a collection } S \text{ s.t. } VC(S) \geq k \} \).

• (20 pts.) Show that \( \text{VC-DIMENSION} \in \Sigma_3 \).

• (10 pts.) Prove that for every \( i \), if \( \Sigma_i = \Pi_i \) then the polynomial hierarchy collapses to the \( i \)-th level.
Problem 4

Recall that \( EXP = DTIME(2^{n^{O(1)}}) \).

- (15 pts.) Prove that if \( P = NP \) then \( \Sigma_k = P \) for all \( k \).
- (15 pts.) Prove that if \( P = NP \) then there is a problem in \( EXP \) that requires circuits of size \( 2^{\Omega(n)} \).

Problem 5

(20 pts.) Prove that if \( NP \subseteq BPP \) then \( NP = RP \).