Chapter 39

Random Walks I

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“A drunk man will find his way home; a drunk bird may wander forever.”

Anonymous,

39.1. Definitions

Let \( G = G(V, E) \) be an undirected connected graph. For \( v \in V \), let \( \Gamma(v) \) denote the set of neighbors of \( v \) in \( G \); that is, \( \Gamma(v) = \{ u \mid vu \in E(G) \} \). A random walk on \( G \) is the following process: Starting from a vertex \( v_0 \), we randomly choose one of the neighbors of \( v_0 \), and set it to be \( v_1 \). We continue in this fashion, in the \( i \)th step choosing \( v_i \), such that \( v_i \in \Gamma(v_{i-1}) \). It would be interesting to investigate the random walk process. Questions of interest include:

(A) How long does it take to arrive from a vertex \( v \) to a vertex \( u \) in \( G \)?

(B) How long does it take to visit all the vertices in the graph.

(C) If we start from an arbitrary vertex \( v_0 \), how long the random walk has to be such that the location of the random walk in the \( i \)th step is uniformly (or near uniformly) distributed on \( V(G) \)?

Example 39.1.1. In the complete graph \( K_n \), visiting all the vertices takes in expectation \( O(n \log n) \) time, as this is the coupon collector problem with \( n - 1 \) coupons. Indeed, the probability we did not visit a specific vertex \( v \) by the \( i \)th step of the random walk is \( (1 - 1/n)^{i-1} \leq e^{-(i-1)/n} \leq 1/n^{10} \), for \( i = \Omega(n \log n) \). As such, with high probability, the random walk visited all the vertex of \( K_n \). Similarly, arriving from \( u \) to \( v \) takes in expectation \( n - 1 \) steps of a random walk, as the probability of visiting \( v \) at every step of the walk is \( p = 1/(n - 1) \), and the length of the walk till we visit \( v \) is a geometric random variable with expectation \( 1/p \).

39.1.1. Walking on grids and lines

Lemma 39.1.2 (Stirling’s formula). For any integer \( n \geq 1 \), it holds \( n! \approx \sqrt{2\pi n} \left( n/e \right)^n \).

39.1.1.1. Walking on the line

Lemma 39.1.3. Consider the infinite random walk on the integer line, starting from 0. Here, the vertices are the integer numbers, and from a vertex \( k \), one walks with probability \( 1/2 \) either to \( k - 1 \) or \( k + 1 \). The expected number of times that such a walk visits 0 is unbounded.

Proof: The probability that in the \( 2i \)th step we visit 0 is \( \frac{1}{2^{2i}} \binom{2i}{i} \). As such, the expected number of times we visit the origin is

\[
\sum_{i=1}^{\infty} \frac{1}{2^{2i}} \binom{2i}{i} \geq \sum_{i=1}^{\infty} \frac{1}{2^{2i^\frac{1}{2}}} = \infty,
\]

since \( \frac{2^{2i}}{2^{\sqrt{2i}}} \leq \binom{2i}{i} \leq \frac{2^{2i}}{\sqrt{2i}} \) [MN98, p. 84]. This can also be verified using the Stirling formula, and the resulting sequence diverges.

\[\square\]

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39.1.1.2. Walking on two dimensional grid

A random walk on the integer grid $\mathbb{Z}^d$, starts from a point of this integer grid, and at each step if it is at point $(i_1, i_2, \ldots, i_d)$, it chooses a coordinate and either increases it by one, or decreases it by one, with equal probability.

**Lemma 39.1.4.** Consider the infinite random walk on the two dimensional integer grid $\mathbb{Z}^2$, starting from $(0,0)$. The expected number of times that such a walk visits the origin is unbounded.

**Proof:** Rotate the grid by 45 degrees, and consider the two new axises $X'$ and $Y'$. Let $x_i$ be the projection of the location of the $i$th step of the random walk on the $X'$-axis, and define $y_i$ in a similar fashion. Clearly, $x_i$ are of the form $j/\sqrt{2}$, where $j$ is an integer. By scaling by a factor of $\sqrt{2}$, consider the resulting random walks $x'_i = \sqrt{2}x_i$ and $y'_i = \sqrt{2}y_i$. Clearly, $x_i$ and $y_i$ are random walks on the integer grid, and furthermore, they are independent. As such, the probability that we visit the origin at the $2i$th step is $P[x'_{2i} = 0 \cap y'_{2i} = 0] = P[x'_{2i} = 0]^2 = \left(\frac{1}{2^i} \left(\frac{2i}{i}\right)^2\right) \geq 1/4i$. We conclude, that the infinite random walk on the grid $\mathbb{Z}^2$ visits the origin in expectation

$$\sum_{i=0}^{\infty} P[x'_i = 0 \cap y'_i = 0] \geq \sum_{i=0}^{\infty} \frac{1}{4i} = \infty,$$

as this sequence diverges. \qed

39.1.1.3. Walking on three dimensional grid

In the following, let $\begin{pmatrix} i \\ a \\ b \\ c \end{pmatrix} = \frac{i!}{a!b!c!}$.

**Lemma 39.1.5.** Consider the infinite random walk on the three dimensional integer grid $\mathbb{Z}^3$, starting from $(0,0,0)$. The expected number of times that such a walk visits the origin is bounded.

**Proof:** The probability of a neighbor of a point $(x, y, z)$ to be the next point in the walk is $1/6$. Assume that we performed a walk for $2i$ steps, and decided to perform $2a$ steps parallel to the $x$-axis, $2b$ steps parallel to the $y$-axis, and $2c$ steps parallel to the $z$-axis, where $a + b + c = i$. Furthermore, the walk on each dimension is balanced, that is we perform $a$ steps to the left on the $x$-axis, and $a$ steps to the right on the $x$-axis. Clearly, this corresponds to the only walks in $2i$ steps that arrives to the origin.

Next, the number of different ways we can perform such a walk is $\frac{(2i)!}{a!b!c!}$, and the probability to perform such a walk, summing over all possible values of $a, b$ and $c$, is

$$\alpha_i = \sum_{a+b+c=i \atop a,b,c \geq 0} \frac{(2i)!}{a!b!c!} \frac{1}{6^{2i}} = \frac{(2i)!}{2^{2i}} \sum_{a+b+c=i \atop a,b,c \geq 0} \left(\frac{i!}{a!b!c!}\right)^2 \left(\frac{1}{3}\right)^i = \left(\frac{2i}{i}\right) \frac{1}{2^{2i}} \sum_{a+b+c=i \atop a,b,c \geq 0} \left(\begin{pmatrix} i \\ a \\ b \\ c \end{pmatrix}\right) \left(\frac{1}{3}\right)^i$$

Consider the case where $i = 3m$. We have that $\begin{pmatrix} i \\ a \\ b \\ c \end{pmatrix} \leq \begin{pmatrix} m \\ m \\ m \end{pmatrix}$. As such,

$$\alpha_i \leq \left(\frac{2i}{i}\right) \frac{1}{2^{2i}} \left(\frac{1}{3}\right)^i \sum_{a+b+c=i \atop a,b,c \geq 0} \left(\begin{pmatrix} i \\ a \\ b \\ c \end{pmatrix}\right) \left(\frac{1}{3}\right)^i = \left(\frac{2i}{i}\right) \frac{1}{2^{2i}} \left(\frac{1}{3}\right)^i \begin{pmatrix} m \\ m \\ m \end{pmatrix}.$$
By the Stirling formula, we have

\[
\binom{i}{m} \approx \frac{\sqrt{2\pi i} (i/e)^i}{(\sqrt{2\pi}/3)^{i/3} \cdot i^{3/2}} = c \frac{3^i}{i},
\]

for some constant \( c \). As such, \( \alpha_i = O\left(\frac{1}{\sqrt{i}} \left(\frac{1}{3}\right)^{i^{3/2}}\right) = O\left(\frac{1}{i^{3/2}}\right) \). Thus,

\[
\sum_{m=1}^{\infty} \alpha_{6m} = O\left(\frac{1}{i^{3/2}}\right) = O(1).
\]

Finally, observe that \( \alpha_{6m} \geq (1/6)^2 \alpha_{6m-2} \) and \( \alpha_{6m} \geq (1/6)^4 \alpha_{6m-4} \). Thus,

\[
\sum_{m=1}^{\infty} \alpha_m = O(1).
\]

39.2. Bibliographical notes

The presentation here follows [Nor98].

References
