Chapter 32

A Bit on Algebraic Graph Theory

By Sariel Har-Peled, March 19, 2024[1]

"The Party told you to reject the evidence of your eyes and ears. It was their final, most essential command."

1984, George Orwell

32.1. Graphs and Eigenvalues

Consider an undirected graph $\mathsf{G} = \mathsf{G}(V, E)$ with n vertices. The adjacency matrix $\mathsf{M}(\mathsf{G})$ of G is the $n \times n$ symmetric matrix where $\mathsf{M}_{ij} = \mathsf{M}_{ji}$ is the number of edges between the vertices v_i and v_j. If G is bipartite, we assume that V is made out of two independent sets X and Y. In this case the matrix $\mathsf{M}(\mathsf{G})$ can be written in block form.

32.1.1. Eigenvalues and eigenvectors

A non-zero vector v is an eigenvector of M, if there is a value λ, known as the ***eigenvalue*** of v, such that $\mathsf{M}v = \lambda v$. That is, the vector v is mapped to zero by the matrix $\mathsf{N} = \mathsf{M} - \lambda I$. This happens only if N is not full ranked, which in turn implies that $\det(\mathsf{N}) = 0$. We have that $f(\lambda) = \det(\mathsf{M} - \lambda I)$ is a polynomial of degree n. It has n roots (not necessarily real), which are the ***eigenvalues*** of M. A matrix $\mathsf{N} \in \mathbb{R}^{n \times n}$ is ***symmetric*** if $\mathsf{N}^T = \mathsf{N}$.

Lemma 32.1.1. *The eigenvalues of a symmetric real matrix $\mathsf{N} \in \mathbb{R}^{n\times n}$ are real numbers.*

Proof: Observe that for any real vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we have that $\sum_{i=1}^n v_i^2 = \langle v, v \rangle \geq 0$. As such, for a vector v with eigenvalue λ, we have

$$0 \leq \langle \mathsf{N}v, \mathsf{N}v \rangle = (\mathsf{N}v)^T \mathsf{N}v = (\lambda v)^T \lambda v = \lambda^2 \langle v, v \rangle .$$

Namely, λ^2 is a non-negative number, which implies that the λ is a real number. ■

Lemma 32.1.2. *Let $\mathsf{N} \in \mathbb{R}^{n\times n}$ be a matrix. Consider two eigenvectors v_1, v_2 that corresponds to two eigenvalues λ_1, λ_2, where $\lambda_1 \neq \lambda_2$. Then v_1 and v_2 are orthogonal.*

Proof: Indeed, $v_1^T \mathsf{N} v_2 = \lambda_2 v_1^T v_2$. Similarly, we have $v_1^T \mathsf{N} v_2 = (\mathsf{N}^T v_1)^T v_2) = \lambda_1 v_1^T v_2$. We conclude that either $\lambda_1 = \lambda_2$, or v_1 and v_2 are orthogonal (i.e., $v_1^T v_2 = 0$). ■

32.1.2. Eigenvalues and eigenvectors of a graph

Since $\mathsf{N} = \mathsf{M}(\mathsf{G})$ the adjacency matrix of an undirected graph is symmetric, all its eigenvalues exists and are real numbers $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$, and their corresponding orthonormal basis vectors are $e_1, \ldots, e_n$.

We will need the following theorem.

Theorem 32.1.3 (Fundamental theorem of algebraic graph theory). *Let* $\mathsf{G} = \mathsf{G}(V, E)$ *be an undirected (multi)graph with maximum degree* d *and with* n *vertices. Let* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ *be the eigenvalues of* $\mathsf{M}(\mathsf{G})$ *and the corresponding orthonormal eigenvectors are* $e_1, \ldots, e_n$. *The following holds.*

(i) *If* G *is connected then* $\lambda_2 < \lambda_1$.

(ii) *For* $i = 1, \ldots, n$, *we have* $|\lambda_i| \leq d$.

(iii) d *is an eigenvalue if and only if* G *is regular.*

(iv) *If* G *is* d*-regular then the eigenvalue* $\lambda_1 = d$ *has the eigenvector* $e_1 = \frac{1}{\sqrt{n}}(1, 1, 1, \ldots, 1)$.

(v) *The graph* G *is bipartite if and only if for every eigenvalue* λ *there is an eigenvalue* $-\lambda$ *of the same multiplicity.*

(vi) *Suppose that* G *is connected. Then* G *is bipartite if and only if* $-\lambda_1$ *is an eigenvalue.*

(vii) *If* G *is* d*-regular and bipartite, then* $\lambda_n = d$ *and* $e_n = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1, -1, \ldots, -1)$, *where there are equal numbers of* 1*s and* -1*s in* e_n.

32.2. Bibliographical Notes

A nice survey of algebraic graph theory appears in [Wes01] and in [Bol98].

References

[Bol98] B. Bollobas. *Modern Graph Theory*. Springer-Verlag, 1998.

[Wes01] D. B. West. *Intorudction to Graph Theory*. 2ed. Prentice Hall, 2001.