Chapter 17

Independent set – Turán's theorem

I don't know why it should be, I am sure; but the sight of another man asleep in bed when I am up, maddens me. It seems to me so shocking to see the precious hours of a man's life - the priceless moments that will never come back to him again - being wasted in mere brutish sleep.

Jerome K. Jerome, Three men in a boat

By Sariel Har-Peled, March 19, 2024[①]

17.1. Turán's theorem

17.1.1. Some silly helper lemmas

We will need the following well-known inequality.

Lemma 17.1.1 (AM-GM inequality: Arithmetic and geometric means inequality.). *For any* $x_1, \ldots, x_n \geq 0$ *we have* $\dfrac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$.

This inequality readily implies the "inverse" inequality: $\dfrac{1}{\sqrt[n]{x_1 x_2 \cdots x_n}} \geq \dfrac{n}{x_1 + x_2 + \cdots + x_n}$

Lemma 17.1.2. *Let* $x_1, \ldots, x_n \geq 0$ *be* n *numbers. We have that* $\sum_{i=1}^{n} \frac{1}{x_i} \geq \frac{n}{(\sum_i x_i)/n}$.

Proof: By the SM-GM inequality and then its "inverse" form, we have

$$\frac{\sum_{i=1}^{n} \frac{1}{x_i}}{n} = \frac{1/x_1 + 1/x_2 + \cdots + 1/x_n}{n} \geq \sqrt[n]{(1/x_1)(1/x_2)\cdots(1/x_n)} = \frac{1}{\sqrt[n]{x_1 x_2 \cdots x_n}} \geq \frac{n}{x_1 + x_2 + \cdots + x_n}. \qquad \blacksquare$$

Lemma 17.1.3. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ *be a graph with* n *vertices, and let* d_{G} *be the average degree in the graph. We have that* $\displaystyle\sum_{v \in \mathsf{V}} \frac{1}{1 + d(v)} \geq \frac{n}{1 + d_{\mathsf{G}}}$.

Proof: Let the ith vertex in G be v_i. Set $x_i = 1 + d(v_i)$, for all i. By Lemma 17.1.2, we have

$$\sum_{i=1}^{n} \frac{1}{1 + d(v_i)} = \sum_{i=1}^{n} \frac{1}{x_i} \geq \frac{n}{(\sum_i x_i)/n} = \frac{n}{[\sum_i (1 + d(v_i))]/n} = \frac{n}{1 + d_{\mathsf{G}}}. \qquad \blacksquare$$

17.1.2. Statement and proof

Theorem 17.1.4 (Turán's theorem). *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ *be a graph with n vertices. The graph* G *has an independent set of size at least* $\dfrac{n}{1 + d_G}$*, where* d_G *is the average vertex degree in G.*

Proof: Let $\pi = (\pi_1, \ldots, \pi_n)$ be a random permutation of the vertices of G. Pick the vertex π_i into the independent set if none of its neighbors appear before it in π. Clearly, v appears in the independent set if and only if it appears in the permutation before all its $d(v)$ neighbors. The probability for this is $1/(1 + d(v))$. Thus, the expected size of the independent set is (exactly)

$$\tau = \sum_{v \in V} \frac{1}{1 + d(v)}, \tag{17.1}$$

by linearity of expectations. Thus, by the probabilistic method, there exists an independent set in G of size at least τ. The claim now readily follows from Lemma 17.1.3. ■

17.1.3. An alternative proof of Turán's theorem

Following a post of this write-up on my blog, readers suggested two modifications. We present an alternative proof incorporating both suggestions.

Alternative proof of Theorem 17.1.4: We associate a charge of size $1/(d(v) + 1)$ with each vertex of G. Let $\gamma(G)$ denote the total charge of the vertices of G. We prove, using induction, that there is always an independent set in G of size at least $\gamma(G)$. If G is the empty graph, then the claim trivially holds. Otherwise, assume that it holds if the graph has at most $n - 1$ vertices, and consider the vertex v of lowest degree in G. The total charge of v and its neighbors is

$$\frac{1}{d(v) + 1} + \sum_{uv \in E} \frac{1}{d(u) + 1} \leq \frac{1}{d(v) + 1} + \sum_{uv \in E} \frac{1}{d(v) + 1} = \frac{d(v) + 1}{d(v) + 1} = 1,$$

since $d(u) \geq d(v)$, for all $uv \in E$. Now, consider the graph H resulting from removing v and its neighbors from G. Clearly, $\gamma(H)$ is larger (or equal) to the total charge of the vertices of $V(H)$ in G, as their degree had either decreased (or remained the same). As such, by induction, we have an independent set in H of size at least $\gamma(H)$. Together with v this forms an independent set in G of size at least $\gamma(H) + 1 \geq \gamma(G)$. Implying that there exists an independent set in G of size

$$\tau = \sum_{v \in V} \frac{1}{1 + d(v)}, \tag{17.2}$$

Now, set $x_v = 1 + d(v)$, and observe that

$$(n + 2|E|)\tau = \left(\sum_{v \in V} x_v \right) \left(\sum_{v \in V} \frac{1}{x_v} \right) \geq \left(\sum_{v \in V} \sqrt{x_v} \frac{1}{\sqrt{x_v}} \right)^2 = n^2,$$

using Cauchy-Schwartz inequality. Namely, $\tau \geq \dfrac{n^2}{n + 2|E|} = \dfrac{n}{1 + 2|E|/n} = \dfrac{n}{1 + d_G}$. ■

Lemma 17.1.5 (Cauchy-Schwartz inequality). *For positive numbers* $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$*, we have*

$$\sum_i \alpha_i \beta_i \leq \sqrt{\sum_i \alpha_i^2} \sqrt{\sum_i \beta_i^2}.$$

17.1.4. An algorithm for the weighted case

In the weighted case, we associate weight $w(v)$ with each vertex of G, and we are interested in the maximum weight independent set in G. Deploying the algorithm described in the first proof of Theorem 17.1.4, implies the following.

Lemma 17.1.6. *The graph $G = (V, E)$ has an independent set of size* $\geq \sum_{v \in V} \frac{w(v)}{1 + d(v)}$.

Proof: By linearity of expectations, we have that the expected weight of the independent set computed is equal to

$$\sum_{v \in V} w(v) \cdot \mathbb{P}[v \text{ in the independent set}] = \sum_{v \in V} \frac{w(v)}{1 + d(v)},$$

■

References

[MR95] R. Motwani and P. Raghavan. *Randomized Algorithms*. Cambridge, UK: Cambridge University Press, 1995.