HW 5

Due on **Monday, April 15, 2024 at 10am**

CS 574: Randomized Algorithms, Spring 2024 Version: **1.0**

Submission guidelines same as previous homework.

13 (100 PTS.) Solving kSAT.

13.A. Using Stirling's formula, give a tight estimate to $\binom{km}{m}$.

13.B. Consider the random walk on the integers that starts at j, and with probability $1/k$ it goes left (i.e., decreases by one), and with probability $(k-1)/k$ it goes right (i.e., increases by 1). The random walk stops once it arrives to 0. Provide a lower bound to the probability that after $2j$ steps (or $3j$ steps, if this is easier or better), the random walk is at zero.

13.C. In the spirit of the randomized algorithm for 2SAT, provide an algorithm for solving a kSAT. Here, you are given a formula with n variables, and every clause has exactly k variables in it.

The algorithm works as follows - it starts with a random assignment, and then performs a random walk for $2n$ (or $3n$) steps on the random assignment (i.e., find an unsatisfied clause, randomly flip one of its variables, repeat). If it found a satisfying assignment it stops. Otherwise, it restarts the walk.

What is the expected running time of the algorithm? (Assuming there is at least one satisfying assignment.) In particular, what is the running time for $k = 3$. Why is this interesting?

14 (100 PTS.) Total walk.

Consider an urn U with n balls, where αn of them are red, and $(1-\alpha)n$ are blue. Assume α is a small constant (say 0.01). Consider the following process: In the ith step, you randomly pick a set S_i of three balls from the urn. Let C_i be the majority of the colors of balls in S_i. You next put the balls of S_i back into U, randomly throw away a ball from U, and add to U a new ball of color C_i. The game is repeated till all the balls are of the same color.

Give an upper and lower bounds on the number of rounds you have to play this game till all balls have the same color. (The bounds should be reasonably tight.)

15 (100 PTS.) Random walks.

15.A. (50 PTS.) Show that the expected time for a random walk to visit every vertex of a strongly connected directed graph is not bounded above by any polynomial function of n. the number of vertices. In other words. construct a directed graph that is strongly connected and where the expected cover time is super- polynomial.

15.B. (50 PTS.) (Harder.) Consider a graph G over $[\![n]\!] = \{1, \ldots, n\}$. A graph G is ***straight*** for $i, j \in [\![n]\!]$, with $i < j$, if there is a path $i_1 i_2 \ldots i_k$ in G between i and j, such that $i = i_1 < i_2 < \ldots < i_k = j$, and $i_t i_{t+1} \in \mathsf{E}(\mathsf{G})$, for all t.

Prove, that there exists a graph G over $[\![n]\!]$, such that for all sets $D \subseteq [\![n]\!]$ of $n/4$ vertices, there exists a set $Y \subseteq [\![n]\!] \setminus D$ of size $\geq n/2$, such that $\mathsf{G} - D$ (i.e., the graph G after we delete all the vertices of D from it) is straight for all $i, j \in Y$.

For credit, your graph G should have as few edges as possible.

(Hint: Use constant degree expanders.)