HW 3

Due on **Monday, March 18, 2024 at 10am**

CS 574: Randomized Algorithms, Spring 2024 Version: **1.0**

Instructions: As in first homework.

7 (100 PTS.) Lazy choices.

Question 17.2 from [MU17].

Consider the following variant of the balanced allocation paradigm: n balls are placed sequentially in n bins, with the bins labeled from 0 to $n-1$. Each ball chooses a bin i uniformly at random, and the ball is placed in the least loaded of bins

$$i, \quad (i+1) \bmod n, \quad (i+2) \bmod n, \quad \ldots, \quad (i+d-1) \bmod n.$$

Argue that, when d is a constant, the maximum load grows as $\Theta(\log n/\log\log n)$. That is, the power of two chocies does not yield an $O(\log\log n)$ result in this case.

8 (100 PTS.) Parallel choices.

Question 17.6 from [MU17].

Consider a parallel version of the balanced allocation paradigm in which we have n/k rounds, where k new balls arrive in each round. Each ball is placed in the least loaded of its d choices, where in this setting the load of each bin is the load at the end of the previous round. Ties are broken randomly. Note that the k new balls cannot affect each other's placement. Give an upper bound on the maximum load as a function of n, d, and k.

9 (100 PTS.) Birthday with choices.

Question 17.14 from [MU17].

The birthday shows that if balls are sequentially thrown randomly into n bins, with constant probability there will be a collision after $\Theta(\sqrt{n})$ balls are thrown.

9.A. (50 PTS.) Suppose that balls are placed sequentially, each ball has two choices of where to be placed, and a ball will choose a bin that avoids a collision if that is possible. Show that there are constants c_1 and c_2 so that after $c_1 n^{2/3} - o(n^{2/3})$ balls are thrown no collision has occurred with probability at least $1/2$, and after $c_2 n^{2/3} + o(n^{2/3})$ balls are thrown at least one collision has occurred with probability at least $1/2$.

9.B. (10 PTS.) How close can you make the constants c_1 and c_2?

9.C. (30 PTS.) Extend your analysis to more than two choices. Specifically, show that if each ball has k choices for some constant k, there are constants $c_{1,k}$ and $c_{2,k}$ so that after $c_{1,k} n^{1-1/k} - o(n61 - 1/k)$ balls are thrown no collision has occurred with probability at least $1/2$, and after $c_{2,k} n^{1-1/k} + o(n^{1-1/k})$ balls are thrown at least one collision has occurred with probability at least $1/2$.

9.D. (10 PTS.) How close can you make the constants $c_{1,k}$ and $c_{2,k}$?

References

[MU17] M. Mitzenmacher and E. Upfal. *Probability and Computing: Randomization and Probabilistic Techniques in Algorithms and Data Analysis*. Cambridge University Press, 2017.