CS 574: Randomized Algorithms

Lecture 9. Lovász Local Lemma and more on Random k-SAT

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- Lovász Local lemma is extending the above when there are dependencies among A_i. (Looking for a needle in a haystack?).

The LLL

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Lemma

(Lovász Local Lemma) Let $A_1, \cdots, A_n$ be a set of "bad" events with $Pr[A_i] \leq p < 1$, and each event A_i is mutually independent of all but at most d of the other A_j. If $e \cdot p(d+1) \leq 1$ (or, also, $4pd \leq 1$), then

$$Pr[\cap_{i=1}^{n} \neg A_i] > 0$$

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Proposition

Suppose that $Z_1, \cdots, Z_m$ is a sequence of independent events and suppose that each A_i is completely determined by some subset $S_i \subseteq \{Z_j\}$. If $S_i \cap S_j = \emptyset$ for $j = j_1, \cdots, j_k$, then A_i is mutually independent of $\{A_{j_1}, \cdots, A_{j_k}\}$.

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Next we see the proof of LLL.

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- Each pair of users $i = 1, \cdots, n$ can choose a path from a collection of m paths F_i.
- Show the following.
- **Class Assignment:** If any path in F_i shares edges with no more than k paths in F_j for all $i \neq j$, then there is a way to choose n edge-disjoint paths connecting the n pairs, provided that $8nk/m \leq 1$.

Lemma

(General Lovász Local Lemma) Let $A_1, \cdots, A_n$ *be a set of "bad" events and let* $D_i \subseteq \{A_1, \cdots, A_n\}$ *be the dependency set of* A_i*. If there exists a set of real numbers* $x_1, \cdots .x_n \in [0, 1)$ *such that* $Pr[A_i] \leq x_i \prod_{j \in D_i}(1 - x_j)$ *for all* i*, then*

$$Pr[\cap_{i=1}^{n} \neg A_i] \geq \prod_{i=1}^{n}(1 - x_i) > 0$$