CS 574: Randomized Algorithms

Lecture 6. Expander Graphs

September 10, 2015

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- **Class assignment:** Show that the diameter of an expander is $O(\log n)$.

What are expander graphs ?

There are three main perspectives of expansion

- Combinatorial ("small" sets have "large" boundaries)
- Linear Algebraic (large spectral gap)
- Probabilistic (random walks converge rapidly)

(One of) The combinatorial definitions we just saw

Definition

A graph G = (V,E) is said to be ϵ - **edge expanding** if for all subsets S of V of size $\leq |V|/2$, the number of cross edges $(e(S, V \setminus S))$ is large. That is,

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In this sense the edge expansion $h(G)$ of a graph is defined as

$$h(G) = min_{S \in V, |S| \leq |V|/2} \frac{e(S, V \backslash S)}{|S|}$$

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- $d - \lambda_2$ is referred to as the spectral gap.

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Theorem (Cheeger's Inequality)

Let G be a d-regular graph with spectrum as defined above. Then

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- Cheeger's says the further away the gap is from zero, the more connected the graph is.

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- For one time step: $p_{t+1}(u) = \sum_{v:(u,v)\in E} \frac{w(u,v)}{d(u)} p_t(v)$.
- In other words, $p_{t+1} = D^{-1}Ap_t = Wp_t$, and for d-regular graphs $p_{t+1} = \frac{1}{d}Ap_t$, $W = \frac{1}{d}A$.

Random Walks on Expanders

Theorem

For all a if $p_0 = \chi_a$ *then*

$$\|p_t - \mathbf{1}/n\| \leq \lambda_2^t$$

Where λ_2 *is the second largest eigenvalue of* $W = D^{-1}A$.

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In about $\log n$ steps of R.W on expander, the distribution is almost uniform.

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Essentially expansion is good and we seek ways of achieving high expansion efficiently

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Theorem (Alon-Bopanna)

For a d-regular graph G

$$\lambda_2 \geq 2(\sqrt{d-1}) - o_n(1)$$

The term $o_n(1)$ goes to zero as $n \to \infty$

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Do such graphs exist with arbitrarily large size?

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- Friedman suggested building expanders by "lifting" the original graph