CS 574: Randomized Algorithms

Lecture 3. The Second Moment Method

September 1, 2015

Threshold Phenomena in Random Graphs

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- Lemma 1 can be shown with first moment, Lemma 2 needs more!

Chebyshev's inequality

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- Define Covariance : $Cov(X, Y) = E(XY) - E(X)E(Y)$.

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- How useful it is depends on variance.
- For independent X_i we have

Lemma

$$Var(X_1 + \cdots, +X_n) = Var(X_1) + \cdots + Var(X_n)$$

Deviation from Mean

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For every $\epsilon > 0$, *the following holds. Let* X_i *i.i.d random variables with* $E[X_i] = \mu$ *and* $Var(X_i) = \sigma^2$. *Let* $X = \frac{X_1 + \cdots + X_n}{n}$ *their empirical mean. If* $n \geq \frac{4}{\epsilon^2}\frac{\sigma^2}{\mu^2}$ *then*

$$Pr[|X - \mu| \geq \epsilon\mu] \leq \frac{\sigma^2}{n\mu^2\epsilon^2} \leq \frac{1}{4}$$

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Claim

If we flip a coin $2k+1$ *times with* $Pr[heads] \geq 3/4$ *then* $Pr[\leq k \text{ flips are heads }] \leq (\frac{3}{4})^k$