Introduction to Markov Chains.

October 29, 2015
A better bound on the Cover Time

Theorem

If $G = (V, E)$ is a connected graph and $R = \max_{x,y \in V} R_{\text{eff}}(x, y)$ is the maximum effective resistance in $G$, then

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Compare the bounds from this to the complete graph and the lollipop.
We saw that a random walk on a connected, non-bipartite graph converges to stationary distribution $P_u(t) \to \frac{d(v)}{2|E|}$.
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- More generally, we define a Markov Chain as a sequence of random variables $(X_t)$ with Markov Property: $Pr[X_t = y | X_{t-1} = x, X_{t-2}...X_0] = Pr[X_t = x | X_{t-1} = y] = P(x, y)$. 
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Markov Chains

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Some Examples

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- Card Shuffling examples.