CS 574: Randomized Algorithms

Lecture 2. The Probabilistic Method and First Moment

August 27, 2015

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- Notice the use of union bound.

Linearity of Expectation

- If $X_1, \cdots, X_n$ are discrete, real-valued r.v then $E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$. No need to know anything about distribution or dependence structure!

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- **Class Assignment 1:**

Claim

In any graph G there exists a cut that cuts at least half the edges.

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- **Class Assignment 2:** Prove Markov.

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- Tight for dense graphs.