Lecture 19. Random Walks and Electrical Networks

October 27, 2015
Random walk on $G$ is a process $\{X_t\}$ such that $X_0 = v_0 \in V$ and if $X_t = v$ then $X_{t+1} = w$ with probability $1/\deg_v$ for every neighbor $w$ of $v$. 
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Cover time of the graph is $\text{cov}(G) = \max_{u \in V} \text{cov}_u(G)$. 
\( G(V, E) \) connected, undirected graph. Each edge is a unit resistor. Create a potential difference at two vertices and induce an electrical flow on the graph. Between every two nodes \( u, v \), there is a potential \( \phi_{u,v} \).
Electrical Networks

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- (K1) Flow into every node equals flow out.

Can generalize from unit resistors to arbitrary conductances $c_{uv}$ so that $r_{uv} = \frac{1}{c_{uv}}$.

There is a map $\phi: V \rightarrow \mathbb{R}$ such that $\phi_{u,v} = \phi(u) - \phi(v)$.

Effective resistance $R_{\text{eff}}(u,v)$ is the potential difference required to induce a current of unit flow between $u, v$. 

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CS 574: Randomized Algorithms
**Electrical Networks**

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- (Ohm) Current flowing from $u$ to $v$ on edge $(u, v)$ is precisely $\frac{\phi_{u,v}}{r_{uv}}$, where $r_{uv}$ is the resistance of $u, v$. ($V = iR$).
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- Can generalize from unit resistors to arbitrary conductances \(c_{uv}\) so that \(r_{uv} = 1/c_{uv}\).
- There is a map \(\phi : V \rightarrow \mathbb{R}\) such that \(\phi_{u,v} = \phi(u) - \phi(v)\).
- Effective resistance \(R_{\text{eff}}(u, v)\) is the potential difference required to induce a current of unit flow between \(u, v\).
Theorem

If $G(V, E)$ has $m$ edges, then for every two nodes $u, v$ we have $H_{uv} + H_{vu} = 2mR_{\text{eff}}(u, v)$.
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For any connected graph $G(V, E)$ we have $\text{cov}(G) \leq |E|(|V| - 1)$. 