CS 574: Randomized Algorithms

Lecture 19. Random Walks and Electrical Networks

October 27, 2015

Hitting times and Cover times

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- Cover time of the graph is $cov(G) = max_{u \in V} cov_u(G)$.

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- There is a map $\phi : V \to \mathbb{R}$ such that $\phi_{u,v} = \phi(u) - \phi(v)$.
- Effective resistance $R_{eff}(u,v)$ is the potential difference required to induce a current of unit flow between u,v.

Theorem

If $G(V, E)$ has m edges, then for every two nodes u, v we have $H_{uv} + H_{vu} = 2mR_{eff}(u, v)$

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For any connected graph $G(V, E)$ we have $cov(G) \leq |E|(|V| - 1)$.