CS 574: Randomized Algorithms

Lecture 15. Martingales and Applications

October 13, 2015

Azuma's Inequality and Proof

Theorem

For every $L > 0$*, if* $\{X_i\}$ *is a martingale with* $|X_i - X_{i-1}| \leq c_i$*, then for every* $\lambda > 0$ *and every* $n \geq 0$ *we have*

$$P[X_n \geq X_0 + \lambda] \leq e^{-\frac{\lambda^2}{2\sum {c_i}^2}}$$

and

$$P[X_n \geq X_0 - \lambda] \leq e^{-\frac{\lambda^2}{2\sum {c_i}^2}}$$

Azuma's Inequality and Proof

Theorem

For every $L > 0$, *if* $\{X_i\}$ *is a martingale with* $|X_i - X_{i-1}| \le c_i$, *then for every* $\lambda > 0$ *and every* $n \ge 0$ *we have*

$$P[X_n \ge X_0 + \lambda] \le e^{-\frac{\lambda^2}{2\sum {c_i}^2}}$$

and

$$P[X_n \ge X_0 - \lambda] \le e^{-\frac{\lambda^2}{2\sum {c_i}^2}}$$

- We will see the proof next

Azuma's Inequality and Proof

Theorem

For every $L > 0$, if $\{X_i\}$ is a martingale with $|X_i - X_{i-1}| \leq c_i$, then for every $\lambda > 0$ and every $n \geq 0$ we have

$$P[X_n \geq X_0 + \lambda] \leq e^{-\frac{\lambda^2}{2\sum {c_i}^2}}$$

and

$$P[X_n \geq X_0 - \lambda] \leq e^{-\frac{\lambda^2}{2\sum {c_i}^2}}$$

- We will see the proof next

Lemma

Let Y be a random variable such that $Y \in [-1, +1]$ and $E[Y] = 0$. Then for any $t \geq 0$, we have $E[e^{tY}] \leq e^{t^2/2}$.

Azuma's Inequality and Proof

Theorem

For every $L > 0$, if $\{X_i\}$ is a martingale with $|X_i - X_{i-1}| \leq c_i$, then for every $\lambda > 0$ and every $n \geq 0$ we have

$$P[X_n \geq X_0 + \lambda] \leq e^{-\frac{\lambda^2}{2\sum {c_i}^2}}$$

and

$$P[X_n \geq X_0 - \lambda] \leq e^{-\frac{\lambda^2}{2\sum {c_i}^2}}$$

- We will see the proof next

Lemma

Let Y be a random variable such that $Y \in [-1, +1]$ and $E[Y] = 0$. Then for any $t \geq 0$, we have $E[e^{tY}] \leq e^{t^2/2}$.

- Can also show something similar for $|X_i - X_{i-1}| \in [a_i, b_i]$.

Concentration of the Chromatic Number of Random Graphs

We will use the vertex-exposure martingale and Azuma's inequality to show sharp concentration of the chromatic number of $G_{n,p}$ around its mean.

Concentration of the Chromatic Number of Random Graphs

We will use the vertex-exposure martingale and Azuma's inequality to show sharp concentration of the chromatic number of $G_{n,p}$ around its mean.

Theorem

(Shamir and Spencer) Let $\chi(G)$ be the chromatic number of $G \in G_{n,p}$.

$$Pr[\chi(G) - E[\chi(G)] \geq \lambda] \leq 2exp(-\frac{\lambda^2}{2n})$$

A Tighter Bound on Chromatic Number

- Note that we did not know $E[\chi(G)]$. We will see next that $E[\chi(G)] \geq \frac{n}{2\log_{p/(1-p)} n}$.

A Tighter Bound on Chromatic Number

- Note that we did not know $E[\chi(G)]$. We will see next that $E[\chi(G)] \geq \frac{n}{2\log_{p/(1-p)} n}$.
- This implies that the expectation is tightly concentrated around it mean, since $E[\chi(G)] >> \sqrt{n}$.

A Tighter Bound on Chromatic Number

- Note that we did not know $E[\chi(G)]$. We will see next that $E[\chi(G)] \geq \frac{n}{2\log_{p/(1-p)} n}$.
- This implies that the expectation is tightly concentrated around it mean, since $E[\chi(G)] >> \sqrt{n}$.
- Easy lower-bound follows from the largest clique size of random graph.

A Tighter Bound on Chromatic Number

- Note that we did not know $E[\chi(G)]$. We will see next that $E[\chi(G)] \geq \frac{n}{2 \log_{p/(1-p)} n}$.
- This implies that the expectation is tightly concentrated around it mean, since $E[\chi(G)] >> \sqrt{n}$.
- Easy lower-bound follows from the largest clique size of random graph.
- Upper-bound much harder.

A Tighter Bound on Chromatic Number

- Note that we did not know $E[\chi(G)]$. We will see next that $E[\chi(G)] \geq \frac{n}{2\log_{p/(1-p)} n}$.
- This implies that the expectation is tightly concentrated around it mean, since $E[\chi(G)] >> \sqrt{n}$.
- Easy lower-bound follows from the largest clique size of random graph.
- Upper-bound much harder.
- We also show a tighter concentration: Chromatic number is concentrated in 4 values w.h.p!