CS 574: Randomized Algorithms

Lecture 14. Introduction to Martingales

October 8, 2015

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A sequence of r.v.'s $X_1, X_2 \cdots$ is called a discrete time martingale, if $E[X_{i+1}|X_0, X_1, \cdots, X_i] = X_i$, for every $i = 0, 1, 2, \cdots$.

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- Equivalently, $E[X_{i+1} - X_i|Y_0, \cdots, Y_i] = 0$ if the set of $Y_0, \cdots, Y_i$ is all the information up to time i. Namely, the difference $X_{i+1} - X_i$ is unbiased on the past up to time i.

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Doob Martingales

- Classic example of a Gambler whose bank roll is X_0. At each time, she chooses to play some game in the casino at some stakes. If we assume that every game is fair (expected utility of playing is 0), but games need not be independent and stakes need not be independent, then the sequence $X_0, X_1, \dots$ is a martingale, where X_i is the amount of money she has at time i.

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- Define Doob Martingale: Let $X_0, X_1, \cdots$ be a sequence or r.v.s. Let Y be also an r.v. with $E[Y] < \infty$. Then $Z_i = E[Y|X_0, X_1, \cdots, X_i]$ is a Doob Martingale.
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- Doob martingales try to estimate function Y with finer and finer estimates.
- Frequently, in application we have $Y = f(Z_1, ..., Z_n)$. In this case, $Z_0 = E(Y)$ and $Z_n = E(Y|Z_1, ..., Z_n) = Y$.

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We say that the martingale $\{X_i\}$ has L-bounded increments if $|X_{i+1} - X_i| \leq L$ for every i.

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Theorem

For every $L > 0$, if $\{X_i\}$ is a martingale with L-bounded increments, then for every $\lambda > 0$ and every $n \geq 0$ we have

$$P[X_n \geq X_0 + \lambda] \leq e^{-\frac{\lambda^2}{2L^2 n}}$$

and

$$P[X_n \geq X_0 - \lambda] \leq e^{-\frac{\lambda^2}{2L^2 n}}$$

Class Assignment: Show the special case for independent r.v.s:

Corollary

If Z_i are independent r.v.s taking values in $[-L, L]$, $Z = \sum Z_i$ and $\mu = E(Z)$, then for every $\lambda > 0$ we have

$$P[Z \geq \mu + \lambda] \leq e^{-\frac{\lambda^2}{2L^2 n}}$$

and

$$P[Z \geq \mu - \lambda] \leq e^{-\frac{\lambda^2}{2L^2 n}}$$

Lipschitz condition and Application to Balls in Bins

- Function $f(z_1, z_2, ..., z_n)$ is L-Lipschitz is changing any one coordinate changes the value of f by at most c in absolute value.

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- Apply Azuma to balls in bins for concentration of the number of empty bins.