CS 574: Randomized Algorithms

Lecture 13. Chernoff Bounds: Another Application to Routing and Introduction to Martingales

October 6, 2015

Chernoff Bound and Randomized Routing

Theorem

(Chernoff Bound) Let X_i be independent bernoulli r.v.s (not necessarily identical), with $E[X_i] = p_i$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = \sum_{i=1}^{n} p_i$ then for every $\beta \geq 1$ we have

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We will use the above to show an efficient, randomized oblivious rounding strategy for rounding on the Hypercube graph.

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There is a randomized, oblivious routing strategy that terminates in $O(n)$ steps with very high probability.

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Definition

A sequence of r.v.'s $X_1, X_2 \cdots$ is called a discrete time martingale, if $E[X_{i+1}|X_0, X_1, \cdots, X_i] = X_i$, for every $i = 0, 1, 2, \cdots$.

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- Equivalently, $E[X_{i+1} - X_i|Y_0, \cdots, Y_i] = 0$ if the set of $Y_0, \cdots, Y_i$ is all the information up to time i. Namely, the difference $X_{i+1} - X_i$ is unbiased on the past up to time i.

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Doob Martingales

- Classic example of a Gambler whose bank roll is X_0. At each time, she chooses to play some game in the casino at some stakes. If we assume that every game is fair (expected utility of playing is 0), then the sequence $X_0, X_1, \ldots$ is a martingale, where X_i is the amount of money she has at time i.

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- Doob martingales try to estimate function Y with finer and finer estimates.