CS 574: Randomized Algorithms

Lecture 11. Chernoff Bounds: An Application to Routing

September 29, 2015

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Theorem

(Chernoff Bound) Let X_i be independent bernoulli r.v.s (not necessarily identical), with $E[X_i] = p_i$. Let $x = \sum_{i=1}^{n} X_i$ and $\mu = \sum_{i=1}^{n} p_i$ then for every $\beta \geq 1$ we have

$$Pr[X \geq \beta\mu] \leq \left(\frac{e^{\beta-1}}{\beta^{\beta}}\right)^{\mu}$$

and

$$Pr[X \leq \frac{\mu}{\beta}] \leq \left(\frac{e^{1/\beta-1}}{\beta^{\beta}}\right)^{\mu}$$

Large Deviations

If we set $\beta = 1 + \delta$ then we get slightly weaker but more simple bounds:

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$$Pr[X \geq (1+\delta)\mu] \leq e^{-\frac{\delta^2 \mu}{3}}$$

and

$$Pr[X \leq (1-\delta)\mu] \leq e^{-\frac{\delta^2 \mu}{2}}$$

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$$opt(\mathcal{J}) = \min\{\max_{e \in E} \#\{i : e \in \gamma_i\}\}$$

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- Since NP-hard, what about approximation (find a set of paths such that the congestion is at most $\alpha \cdot OPT$)?

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Theorem

Let $n = |V|$ and suppose that $n \geq 4$. If there is a multi-flow F that routes the demands (s_i, t_i) then there is an integral flow F' that routes the demands and

$$\max_{e \in E} F'(e) \leq C \frac{\log n}{\log \log n}(1 + \max_{e \in E} F(e))$$

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Class Assignment: Write the (poly-sized) LP for the multi-flow above.