

# CS 574: Randomized Algorithms

## Lecture 1. Introduction to Randomness

August 25, 2015

# Administrativa

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- Class webpage:  
<https://courses.engr.illinois.edu/cs574/fa2015/>

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# Tentative Syllabus

**Weeks 1-6, Discrete Probability:** First and Second Moment method, coupon collector problem, Probabilistic Method, Chernoff Bound and applications, Martingales and Azuma. Lovasz Local Lemma, Method of Conditional Probabilities (perhaps if time). 2 homework assignments corresponding to those topics.

# Tentative Syllabus

**Weeks 7-9, High-dimensional probability:** Bourgain's embedding, Curse of Dimensionality, Dimension Reduction, Matrix Concentration (Golden-Thompson, Bernstein), Random Graph eigenvalues via matrix concentration, Spectral Graph Sparsification via Sampling. 1 homework assignment.



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**Weeks 10-12, Random Walk topics:** Random Walks: hitting times, cover times etc, Markov Chains and Mixing, Eigenvalues, Expanders and Mixing. 1 homework assignment.

**Remaining time, Special Topics:** Including but not limited to Lifts and expansion, Algorithms for Stochastic Block Models, Random Graph Spectra.

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- There is no fixed textbook, but most of the material we will cover can be found in “Probability and Computing: Randomized Algorithms and Probabilistic Analysis” by Mitzenmacher and Upfal.
- We will have class assignments in almost every class, where you will work in groups and solve a question relevant to the lecture topic. Those will NOT contribute to the grade, it is just for a comprehensive exercise.

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- Final exams: large size of material, choose problems independently.

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- This class: analyze running time, complexity, techniques...

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- One-sided error.

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- How to choose a uniformly random prime?
- Use randomized primality testing algorithm (Miller-Rabin)!

# Pattern Matching

- Given two strings  $X = x_1x_2 \cdots x_n$  and  $Y = y_1y_2 \cdots y_m$ , with  $m < n$  we want to check whether  $Y = X(j)$  for some  $j \in \{1, \dots, n - m + 1\}$ . Here  $X(j) = x_jx_{j+1} \cdots x_{j+m-1}$ .

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- Naive algorithm too slow, there are  $O(n + m)$  deterministic algorithms but we will see a simple random one (Karp-Rabin).



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- Still  $O(nm)$  running time. Can we do something more clever?

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- **Probability Technique: Error Amplification.** Can choose  $k$  independent vectors...