CS 574. Randomized Algorithms
Problem Set 2

Alexandra Kolla

due October 20, 2015

Collaboration Policy: The homework can be worked in groups of up to 3 students each (2 would be optimal, but 1 and 3 are both accepted).

One submission per team is sufficient. Please write the solution for each of the problems on a separate sheet of paper. Write your team names and netids on each submission and please **staple** all the sheets together.

Extra 5% for typed homeworks (preferably pdf).

Homework is due on or before the end of class, October 20. Only one late homework per person will be allowed. If you submit more than one homework late, you will get no grade for the excess late homeworks.

Problem 1 *Independence Matters*

(5 pts.)

Consider a function $f(x_1, ..., x_n)$ which is 1-Lipschitz. Let $X_1, ..., X_n$ be random variables, all falling in the range $[0, 1]$. By definition, we know that for $Y_i = E[f(X_1, ..., X_n)|X_1, ..., X_i]$ the sequence $\{Y_i\}$ is a martingale. Show that if $X_1, ..., X_n$ are not independent then it is no longer (always) true that the $Y_n - Y_0$ is concentrated. That is, you need to give a specific example where Azuma's inequality would break in such a case. Recall that in class we showed that if they are independent then $|Y_{i+1} - Y_i| < 1$, thus Azuma's inequality applies.

Problem 2 *Is Markov's and Chenrnoff's Inequality Tight?*

(10 pts.)

1. For an integer k, define a non-negative random variable X_k, such that $E[X_k] = 1$, and $Pr[X_k \geq k] = 1/k$. Namely, Markov's inequality can be tight for any k.

2. Consider a positive integral random variable X with $\Delta = E[X] < \infty$. Furthermore, for any number x, there exists an integer $y(x) > x$, such that we have $Pr[X \geq x\Delta] \geq Pr[X \geq y\Delta] \geq c/y$, where $c > 0$ is some arbitrary constant. Prove, that no such random variable X exists.

3. Let $X_1, ..., X_n$ be independent random variables taking values from $\{-1, 1\}$ with equal probability. Let $X = \sum X_i$. Let $P(n, \Delta) = Pr[X > \Delta]$. Prove that for $\Delta \leq n/C$ we have $P(n, \Delta) \geq \frac{1}{C} e^{-\frac{\Delta^2}{Cn}}$, where C is a suitable constant. Namely, Chernoff's bound $P(n, \Delta) \leq e^{-\frac{\Delta^2}{2n}}$ is almost tight. [Hint: Use Stirling's Formula]

Problem 3 *Disjoint Cycles*

(5 pts)

Let G be an arbitrary k-regular directed graph (i.e., every vertex has in and out degree k). In this problem, we will show, using the Lov'asz Local Lemma (LLL), that G contains at least $\lfloor \frac{k}{3 \ln k} \rfloor$ vertex-disjoint cycles.

1. Suppose the vertices of G are partitioned into $c = \lfloor \frac{k}{3 \ln k} \rfloor$ components by assigning each vertex to a component chosen independently and u.a.r. For each vertex v, let A_v be the event that v has no edge to another vertex in its component. Show that $Pr[A_v] \leq k^{-3}$.

2. Let D_v denote the "dependency set" of event A_v (i.e., A_v is independent of all events A_u except for those in D_v). Show that $|D_v| \leq (k+1)^2$.

3. Deduce from parts (1) and (2) and the LLL that G contains at least c vertex-disjoint cycles.