CS 574: Randomized Algorithms
Take-Home Final

Alexandra Kolla

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Version: 2.0

Collaboration Policy: There is no collaboration allowed for this exam.

Problem 1 *Random Stuff*

(5 pts.) Prove the following two facts:

1. Let $Z_1, \cdots, Z_k$ be i.i.d. random variables distributed according to the Binomial distribution with parameters n and p. Let $M = \max_i Z_i$. Show that $E[M] = np + O(\sqrt{np \log k})$.

2. Show that any weighted tree metric can be mapped isometrically into $\mathbb{R}^k$ equipped with the ℓ_1 metric.

Problem 2 *Random Graphs*

(5 pts.)

This question involves looking into Poisson approximations which we briefly mentioned in class during the balls-in-bins lectures, but did not cover fully. This is an opportunity to learn the relevant material.

1. Theorem 5.7 (Chapter 5, Probability and Computing book) shows that any event that occurs with small probability in the balls-in-bins setting where the number of balls in each bin is an independent poisson random variable also occurs with small probability in the standard ball-in-bins model. Prove a similar statement for random graphs: Every event that happens with small probability in the $G_{n,p}$ model also happens with small probability in the $G_{n,N}$ model for $N = \binom{n}{2}p$. Here $G_{n,N}$ is the set of graphs with n vertices and exactly N edges.

2. An undirected graph on n vertices is disconnected if there exists a set of $k < n$ vertices such that there is no edge between this set and the rest of the graph. Otherwise, the graph is connected. Show that there exists a constant c such that if $N \geq cn \log n$ then, with probability $1 - o(1)$, a graph chosen randomly from $G_{n,N}$ is connected.

Problem 3 *Dominoes*

(5 pts.)

Suppose that you are arranging a chain of n dominos s that, once you are done, you can have them all fall sequentially in a pleasing manner by knocking down the lead domino. Each time you try to place a domino in the chain, there is some chance that it falls, taking down all of the other dominos you have already carefully placed. In that case, you must start all over gain from the very first domino.

1. Let's call each time you try to place a domino a trial. Each trial succeeds with probability p. Using Wald's equation, find the expected number of trials necessary before your arrangement is ready. Calculate this number of trials for $n = 100$ and $p = 0.1$ for fun.

2. Suppose instead that you can break your arrangement into k components, each of size n/k, in such a way so that once a component is complete, it will not fall when you place further dominos (placing a further domino might take down another later component, but none of the ones that are already complete). Find the expected number of trials necessary before your arrangement is ready in that case. Calculate the number of trials for $n = 100$, $k = 10$ and $p = 0.1$ and compare with your previous result in (1).

Problem 4 *Random Walk on Cycle*

(5 pts.)

Let $C_n = (V, E)$ denote the cycle of length n; that is, the vertices are $V = \{v_1, ..., v_n\}$, and there is an edge between v_i and v_{i+1}, for $i = 1, ..., n-1$, and v_n is connected to v_1. You are starting a random walk from $X_0 = v_1$. At the i-th step, you do one of the following with probability $1/3$

- Stay put (i.e., $X_i = X_{i-1}$).
- Go left (i.e., if $X_{i-1} = v_j$, then $X_i = v_{j-1}$, where $v_0 \equiv v_n$).
- Go right (i.e., $X_{i-1} = v_j$, then $X_i = v_{j+1}$, where $v_{n+1} \equiv v_1$).

Let δ be a parameter, $\delta \in (0, 1)$, and let $v_t \in V$ be an arbitrary vertex. Let $m = m(\delta, n)$ be the minimum number of steps, so that for all $i \geq m$, we have that

$$|Pr[X_i = v_t] - 1/n| \leq \delta/n$$

Provide an upper and lower bound, as tight as possible, on the value of $m(\delta, n)$.