Approximate Max Cut

Lecture 24

November 19, 2014

Part I

Normal distribution

Normal distribution – proof

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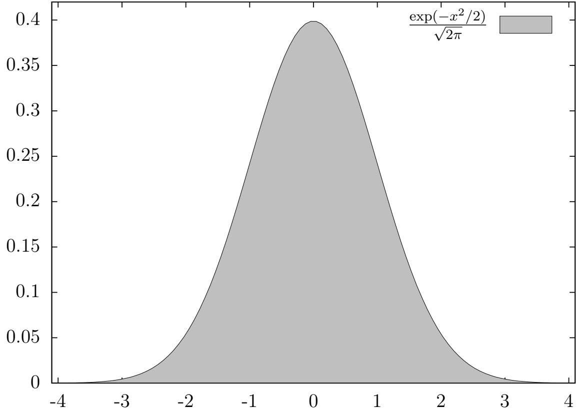
One dimensional normal distribution

1. A random variable X has ***normal distribution*** if $\Pr[X = x] = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$.
2. $X \sim N(0, 1)$.



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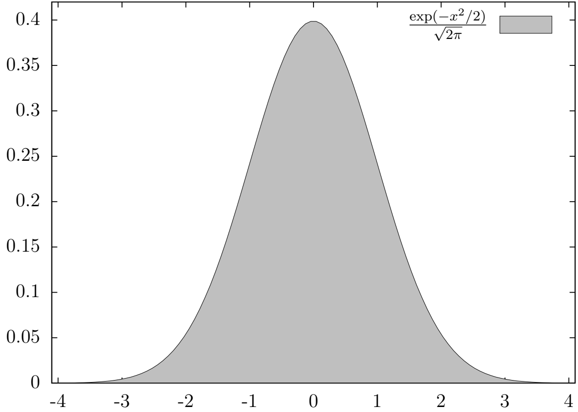

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3. $\mathrm{x} = (x_1, \ldots, x_n)$ has d-dimensional normal distributed (i.e., $\mathrm{v} \sim N^n(0,1)$
   $\iff v_1, \ldots, v_n \sim N(0,1)$
4. $\mathrm{v} \in \mathbb{R}^n$, such that $\|\mathrm{v}\| = 1$.
5. Let $\mathrm{x} \sim N^n(0,1)$. Then $z = \langle \mathrm{v}, \mathrm{x} \rangle$ has...
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Part II

Approximate Max Cut

The movie so far...

Summary: It sucks.

1. Seen: Examples of using rounding techniques for approximation.
2. So far: Relaxed optimization problem is LP.
3. But... We know how to solve ***convex programming***.
4. Convex programming $\gg$ LP.
5. Convex programming can be solved in polynomial time.
6. Solving convex programming is outside scope: assume doable in polynomial time.
7. Today's lecture:
    1. Revisit **MAX CUT**.
    2. Show how to relax it into semi-definite programming problem.
    3. Solve relaxation.
    4. Show how to round the relaxed problem.

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Problem Statement: MAX CUT

Since this is a theory class, we will define our problem.

1. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$: undirected graph.
2. $\forall ij \in \mathbf{E}$: nonnegative weights ω_{ij}.
3. **MAX CUT** (***maximum cut problem***): Compute set $S \subseteq \mathbf{V}$ maximizing weight of edges in cut $\left(S, \overline{S}\right)$.
4. $ij \notin \mathbf{E} \implies \omega_{ij} = 0$.
5. ***weight*** of cut: $w\left(S, \overline{S}\right) = \sum_{i \in S,\ j \in \overline{S}} \omega_{ij}$.
6. Known: problem is **NP-Complete**.
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Max cut as integer program

because what can go wrong?

1. Vertices: $\mathbf{V} = \{1, \ldots, n\}$.
2. ω_{ij}: non-negative weights on edges.
3. max cut $w(S, \overline{S})$ is computed by the integer quadratic program:

$$(Q) \qquad \max \quad \frac{1}{2}\sum_{i<j} \omega_{ij}(1 - y_i y_j)$$

$$\text{subject to:} \quad y_i \in \{-1, 1\} \qquad \forall i \in \mathbf{V}.$$

4. Set: $S = \left\{ i \;\middle|\; y_i = 1 \right\}$.
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Max cut as integer program

because what can go wrong?

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Relaxing $-1, 1$...

Because 1 and -1 are just vectors.

1. Solving quadratic integer programming is of course **NP-Hard**.
2. Want a relaxation...
3. 1 and -1 are just roots of unity.
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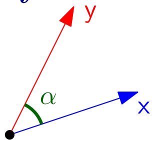

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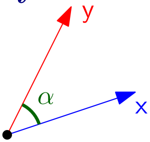

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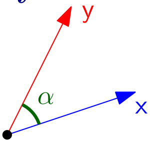

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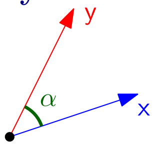

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1. semi-definite programming: special case of convex programming.
2. Can be solved in polynomial time.
3. Solve within a factor of $(1 + \varepsilon)$ of optimal, for any $\varepsilon > 0$, in polynomial time.
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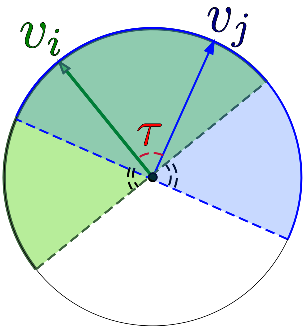
Analysis...

Intuition: with good probability, vectors in the solution of (P) that have large angle between them would be separated by cut.

Lemma

$$\Pr\Big[\mathrm{sign}\Big(\langle v_i, \vec{r}\,\rangle\Big) \neq \mathrm{sign}(\langle v_j, \vec{r}\,\rangle)\Big] = \frac{1}{\pi}\arccos\Big(\langle v_i, v_j\rangle\Big) = \frac{\tau}{\pi}.$$



Proof...

1. Think v_i, v_j and $\vec{r}$ as being in the plane.
2. ... reasonable assumption!
   1. g: plane spanned by v_i and v_j.
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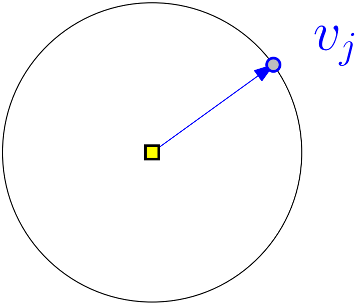

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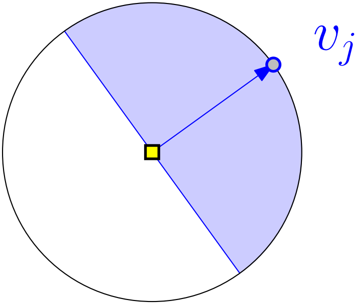

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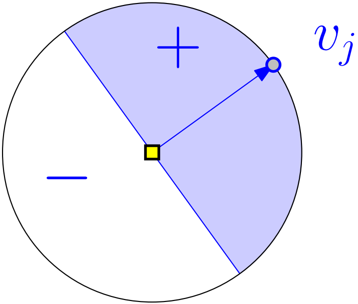

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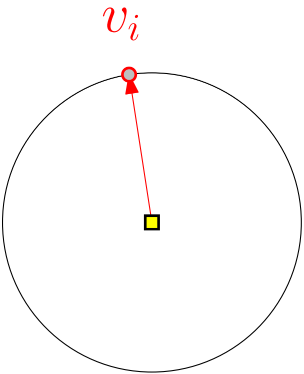

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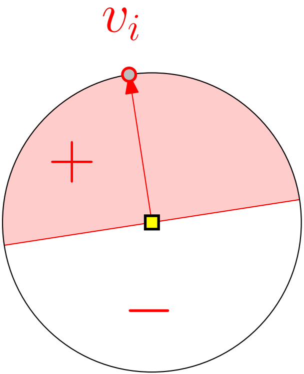

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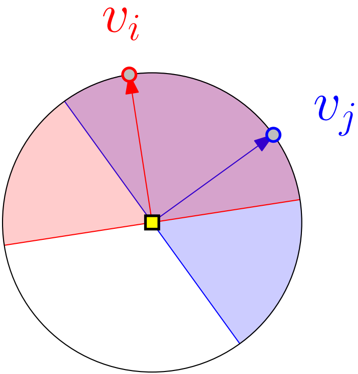

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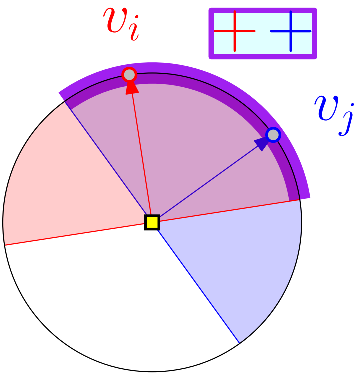

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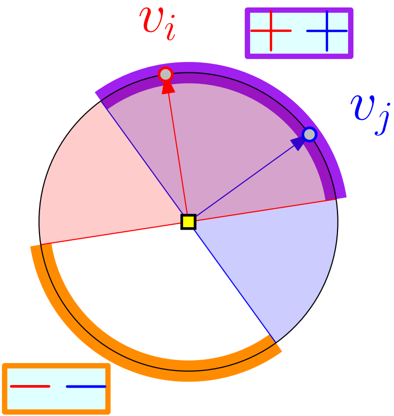

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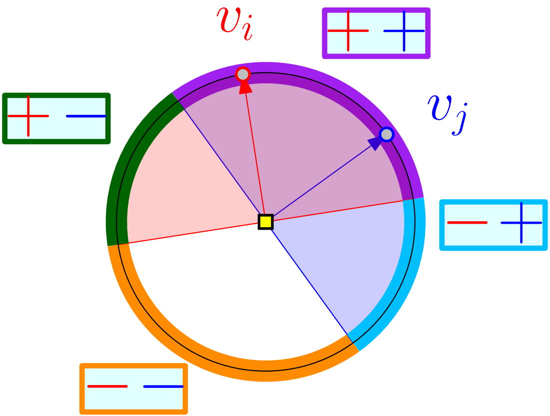

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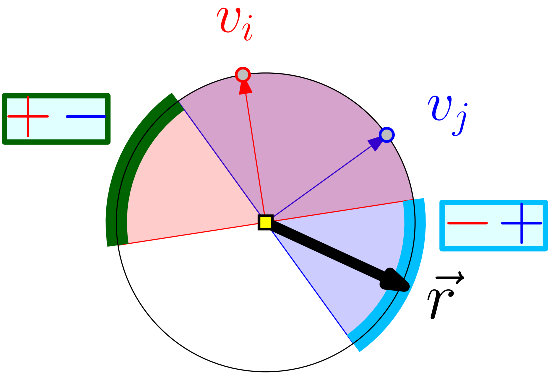

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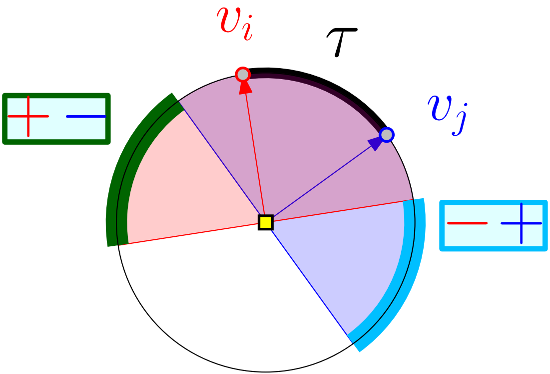
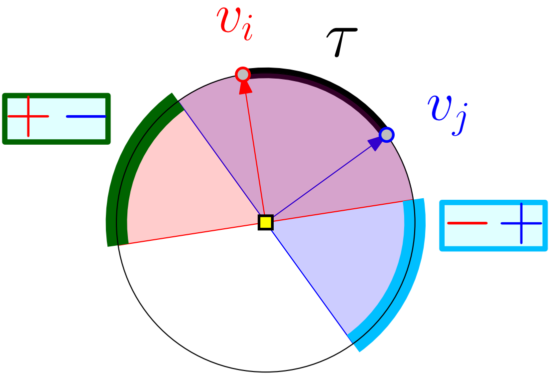


Proof via figure...



$\boldsymbol{\tau} = \mathbf{arccos}\Big(\langle v_i, v_j\rangle\Big)$

Proof...

1. Think v_i, v_j and $\vec{r}$ as being in the plane.
2. $\mathrm{sign}(\langle v_i, \vec{r}\,\rangle) \neq \mathrm{sign}(\langle v_j, \vec{r}\,\rangle)$ happens only if $\vec{r}$ falls in the double wedge formed by the lines perpendicular to v_i and v_j.
3. angle of double wedge $=$ angle τ between v_i and v_j.
4. v_i and v_j are unit vectors: $\langle v_i, v_j\rangle = \cos(\tau)$.
   $\tau = \angle v_i v_j$.
5. Thus,

$$\Pr\Big[\mathrm{sign}(\langle v_i, \vec{r}\,\rangle) \neq \mathrm{sign}(\langle v_j, \vec{r}\,\rangle)\Big] = \frac{2\tau}{2\pi} = \frac{1}{\pi} \cdot \arccos(\langle v_i, v_j\rangle)\,,$$

as claimed. ∎

Theorem

Theorem

Let W *be the random variable which is the weight of the cut generated by the algorithm. We have*

$$\mathbf{E}\Big[W\Big] = \frac{1}{\pi}\sum_{i<j}\omega_{ij}\arccos\Big(\langle v_i, v_j\rangle\Big).$$

Proof

1. X_{ij}: indicator variable $= 1 \iff$ edge ij is in the cut.
2. $\mathbf{E}[X_{ij}] = \Pr\left[\operatorname{sign}(\langle v_i, \vec{r}\,\rangle) \neq \operatorname{sign}(\langle v_j, \vec{r}\,\rangle)\right]$
   $= \frac{1}{\pi}\arccos\Big(\langle v_i, v_j\rangle\Big)$, by lemma.
3. $W = \sum_{i<j} \omega_{ij} X_{ij}$, and by linearity of expectation...

$$\mathbf{E}[W] = \sum_{i<j} \omega_{ij}\,\mathbf{E}[X_{ij}] = \frac{1}{\pi}\sum_{i<j} \omega_{ij} \arccos\Big(\langle v_i, v_j\rangle\Big).$$

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■

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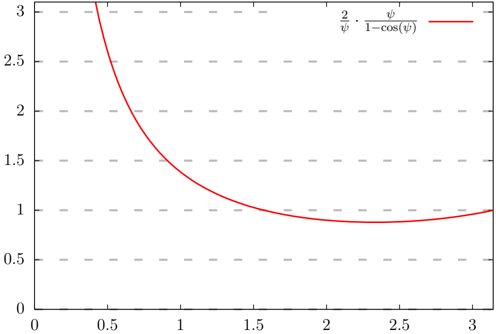
For $-1 \leq y \leq 1$, *we have* $\dfrac{\arccos(y)}{\pi} \geq \alpha \cdot \dfrac{1}{2}(1 - y)$, *where*
$$\alpha = \min_{0 \leq \psi \leq \pi} \frac{2}{\pi} \frac{\psi}{1 - \cos(\psi)}.$$



Lemma restated + proof

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Proof.

1. $y = \cos(\psi)$.
2. Inequality becomes: $\frac{\psi}{\pi} \geq \alpha \frac{1}{2}(1 - \cos\psi)$. Reorganizing,
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Lemma

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$\alpha > 0.87856$.

Proof.

Using simple calculus, one can see that α achieves its value for $\psi = 2.331122...$, the nonzero root of $\cos\psi + \psi\sin\psi = 1$. □

Result

Theorem

The above algorithm computes in expectation a cut with total weight $\alpha \cdot \mathbf{Opt} \geq \mathbf{0.87856Opt}$, *where* $\mathbf{Opt}$ *is the weight of the maximal cut.*

Proof.

Consider the optimal solution to (P), and lets its value be $\gamma \geq \mathbf{Opt}$. By lemma:

$$\mathbf{E}[W] = \frac{1}{\pi} \sum_{i<j} \omega_{ij} \arccos(\langle v_i, v_j \rangle)$$

$$\geq \sum_{i<j} \omega_{ij} \alpha \frac{1}{2}(1 - \langle v_i, v_j \rangle) = \alpha\gamma \geq \alpha \cdot \mathbf{Opt}. \quad \blacksquare$$

$\square$

SDP: Semi-definite programming

1. $x_{ij} = \langle v_i, v_j \rangle$.
2. M: $n \times n$ matrix with x_{ij} as entries.
3. $x_{ii} = 1$, for $i = 1, \ldots, n$.
4. V: matrix having vectors $v_1, \ldots, v_n$ as its columns.
5. $M = V^T V$.
6. $\forall v \in \mathbb{R}^n$: $v^T M v = v^T A^T A v = (Av)^T (Av) \geq 0$.
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Lemma

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Let $\mathcal{U}$ be the set of $n \times n$ positive semidefinite matrices. The set $\mathcal{U}$ is convex.

Proof.

Consider $A, B \in \mathcal{U}$, and observe that for any $t \in [0,1]$, and vector $v \in \mathbb{R}^n$, we have:

$$v^T \Big(tA + (1-t)B \Big)\, v = v^T \Big(tAv + (1-t)Bv \Big)$$
$$= tv^T Av + (1-t)v^T Bv \geq 0 + 0 \geq 0,$$

since A and B are positive semidefinite. □

More on positive semidefinite matrices

1. PSD matrices corresponds to ellipsoids.
2. $x^T A x = 1$: the set of vectors solve this equation is an ellipsoid.
3. Eigenvalues of a PSD are all non-negative real numbers.
4. Given matrix: can in polynomial time decide if it is PSD.
5. ... by computing the eigenvalues of the matrix.
6. $\implies$ SDP: optimize a linear function over a convex domain.
7. SDP can be solved using interior point method, or the ellipsoid method.
8. See **Boyd and Vandenberghe [2004], Grötschel et al. [1993]** for more details.
9. Membership oracle: ability to decide in polynomial time, given a solution, whether its feasible or not.

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Bibliographical Notes

1. Approx. algorithm presented by Goemans and Williamson **Goemans and Williamson [1995]**.
2. **Håstad [2001]** showed that MAX CUT can not be approximated within a factor of $\mathbf{16/17 \approx 0.941176}$.
3. **Khot et al. [2004]** showed a hardness result that matches the constant of Goemans and Williamson (i.e., one can not approximate it better than α, unless $\mathsf{P} = \mathsf{NP}$).

Bibliographical Notes

1. Relies on two conjectures: "Unique Games Conjecture" and "Majority is Stablest".
2. "Majority is Stablest" conjecture was proved by **Mossel et al. [2005]**.
3. Not clear if the "Unique Games Conjecture" is true, see the discussion in **Khot et al. [2004]**.
4. Goemans and Williamson work spurred wide research on using SDP for approximation algorithms.

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