Chapter 23

Entropy, Randomness, and Information

CS 573: Algorithms, Fall 2014
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23.1 Entropy

23.1.0.1 Quote

> "If only once - only once - no matter where, no matter before what audience - I could better the record of the great Rastelli and juggle with thirteen balls, instead of my usual twelve, I would feel that I had truly accomplished something for my country. But I am not getting any younger, and although I am still at the peak of my powers there are moments - why deny it? - when I begin to doubt - and there is a time limit on all of us."
>
> –Romain Gary, The talent scout.

23.2 Entropy

23.2.0.2 Entropy: Definition

Definition 23.2.1. The ***entropy*** in bits of a discrete random variable X is

$$\mathbb{H}(X) = -\sum_{x} \mathbf{Pr}\Big[X = x\Big] \lg \mathbf{Pr}\Big[X = x\Big].$$

Equivalently, $\mathbb{H}(X) = \mathbf{E}\Big[\lg \frac{1}{\mathbf{Pr}[X]}\Big]$.

23.2.0.3 Entropy intuition...

Intuition... $\mathbb{H}(X)$ is the number of ***fair*** coin flips that one gets when getting the value of X.

Interpretation from last lecture... Consider a (huge) string $S = s_1 s_2 \ldots s_n$ formed by picking characters independently according to X. Then

$$|S|\,\mathbb{H}(X) = n\mathbb{H}(X)$$

is the minimum number of bits one needs to store the string S.

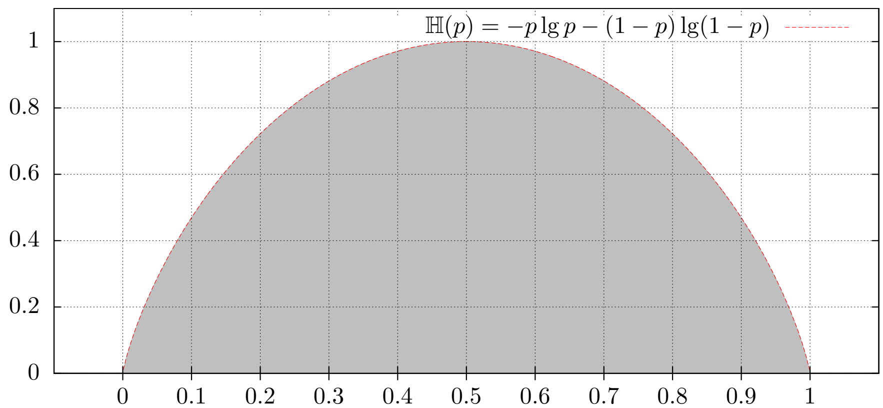
23.2.0.4 Binary entropy

$\mathbb{H}(X) = -\sum_x \mathbf{Pr}\Big[X = x\Big] \lg \mathbf{Pr}\Big[X = x\Big]$
$\implies$

Definition 23.2.2. The ***binary entropy*** function $\mathbb{H}(p)$ for a random binary variable that is 1 with probability p, is $\mathbb{H}(p) = -p \lg p - (1-p)\lg(1-p)$. We define $\mathbb{H}(0) = \mathbb{H}(1) = 0$.

Q: How many truly random bits are there when given the result of flipping a single coin with probability p for heads?

23.2.0.5 Binary entropy: $\mathbb{H}(p) = -p \lg p - (1-p)\lg(1-p)$



(A) $\mathbb{H}(p)$ is a concave symmetric around 1/2 on the interval $[0, 1]$.
(B) maximum at 1/2.
(C) $\mathbb{H}(3/4) \approx 0.8113$ and $\mathbb{H}(7/8) \approx 0.5436$.
(D) $\implies$ coin that has 3/4 probably to be heads have higher amount of "randomness" in it than a coin that has probability 7/8 for heads.

23.2.0.6 And now for some unnecessary math

(A) $\mathbb{H}(p) = -p \lg p - (1-p)\lg(1-p)$
(B) $\mathbb{H}'(p) = -\lg p + \lg(1-p) = \lg \frac{1-p}{p}$
(C) $\mathbb{H}''(p) = \frac{p}{1-p} \cdot \left(-\frac{1}{p^2}\right) = -\frac{1}{p(1-p)}$.
(D) $\implies \mathbb{H}''(p) \leq 0$, for all $p \in (0, 1)$, and the $\mathbb{H}(\cdot)$ is concave.
(E) $\mathbb{H}'(1/2) = 0 \implies \mathbb{H}(1/2) = 1$ max of binary entropy.
(F) $\implies$ balanced coin has the largest amount of randomness in it.

23.2.1 Task at hand: Squeezing good random bits...

23.2.1.1 ...out of bad random bits...

(A) $b_1, \ldots, b_n$: result of n coin flips...
(B) From a faulty coin!
(C) p: probability for head.
(D) We need fair bit coins!
(E) Convert $b_1, \ldots, b_n \implies b'_1, \ldots, b'_m$.
(F) **New bits must be truly random**: Probability for head is 1/2.
(G) **Q:** How many truly random bits can we extract?

23.2.2 Intuitively...

23.2.2.1 Squeezing good random bits out of bad random bits...

Question... Given the result of n coin flips: $b_1, \ldots, b_n$ from a faulty coin, with head with probability p, how many truly random bits can we extract?

If believe intuition about entropy, then this number should be $\approx n\mathbb{H}(p)$.

23.2.2.2 Back to Entropy

(A) ***entropy*** of X is $\mathbb{H}(X) = -\sum_x \mathbf{Pr}\left[X = x\right] \lg \mathbf{Pr}\left[X = x\right]$.
(B) Entropy of uniform variable..

> Example 23.2.3. A random variable X that has probability $1/n$ to be i, for $i = 1, \ldots, n$, has entropy $\mathbb{H}(X) = -\sum_{i=1}^{n} \frac{1}{n} \lg \frac{1}{n} = \lg n$.

(C) Entropy is oblivious to the exact values random variable can have.
(D) $\implies$ random variables over $-1, +1$ with equal probability has the same entropy (i.e., 1) as a fair coin.

23.2.2.3 Lemma: Entropy additive for independent variables

23.2.2.4 Lemma: Entropy additive for independent variables

Lemma 23.2.4. *Let X and Y be two independent random variables, and let Z be the random variable (X, Y). Then $\mathbb{H}(Z) = \mathbb{H}(X) + \mathbb{H}(Y)$.*

23.2.2.5 Proof

In the following, summation are over all possible values that the variables can have. By the independence of X and Y we have

$$\begin{aligned}
\mathbb{H}(Z) &= \sum_{x,y} \mathbf{Pr}\left[(X,Y) = (x,y)\right] \lg \frac{1}{\mathbf{Pr}[(X,Y) = (x,y)]} \\
&= \sum_{x,y} \mathbf{Pr}\left[X = x\right] \mathbf{Pr}\left[Y = y\right] \lg \frac{1}{\mathbf{Pr}[X = x]\,\mathbf{Pr}[Y = y]} \\
&= \sum_{x} \sum_{y} \mathbf{Pr}[X = x]\,\mathbf{Pr}[Y = y] \lg \frac{1}{\mathbf{Pr}[X = x]} \\
&\quad + \sum_{y} \sum_{x} \mathbf{Pr}[X = x]\,\mathbf{Pr}[Y = y] \lg \frac{1}{\mathbf{Pr}[Y = y]}
\end{aligned}$$

23.2.2.6 Proof continued

$$\begin{aligned}\mathbb{H}(Z) &= \sum_x \sum_y \mathbf{Pr}[X = x]\,\mathbf{Pr}[Y = y] \lg \frac{1}{\mathbf{Pr}[X = x]} \\ &\quad + \sum_y \sum_x \mathbf{Pr}[X = x]\,\mathbf{Pr}[Y = y] \lg \frac{1}{\mathbf{Pr}[Y = y]} \\ &= \sum_x \mathbf{Pr}[X = x] \lg \frac{1}{\mathbf{Pr}[X = x]} \\ &\quad + \sum_y \mathbf{Pr}[Y = y] \lg \frac{1}{\mathbf{Pr}[Y = y]} \\ &= \mathbb{H}(X) + \mathbb{H}(Y)\,.\end{aligned}$$

■

23.2.2.7 Bounding the binomial coefficient using entropy
23.2.2.8 Bounding the binomial coefficient using entropy

Lemma 23.2.5. $q \in [0, 1]$
nq is integer in the range $[0, n]$.
Then

$$\frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{nq} \leq 2^{n\mathbb{H}(q)}.$$

23.2.2.9 Proof

Holds if $q = 0$ or $q = 1$, so assume $0 < q < 1$. We have

$$\binom{n}{nq} q^{nq} (1-q)^{n-nq} \leq (q + (1-q))^n = 1.$$

We also have: $q^{-nq}(1-q)^{-(1-q)n} = 2^{n(-q\lg q - (1-q)\lg(1-q))} = 2^{n\mathbb{H}(q)}$, we have

$$\binom{n}{nq} \leq q^{-nq}(1-q)^{-(1-q)n} = 2^{n\mathbb{H}(q)}.$$

23.2.3 Proof continued

23.2.3.1 Other direction...

(A) $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
(B) $\sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} = \sum_{i=0}^{n} \mu(i)$.
(C) Claim: $\mu(nq) = \binom{n}{nq} q^{nq}(1-q)^{n-nq}$ largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
(D) $\Delta_k = \mu(k) - \mu(k+1) = \binom{n}{k} q^k (1-q)^{n-k} \left(1 - \frac{n-k}{k+1}\frac{q}{1-q}\right)$,
(E) sign of Δ_k = size of last term...
(F) $\text{sign}(\Delta_k) = \text{sign}\left(1 - \frac{(n-k)q}{(k+1)(1-q)}\right)$
$= \text{sign}\left(\frac{(k+1)(1-q) - (n-k)q}{(k+1)(1-q)}\right)$.

23.2.3.2 Proof continued

(A) $(k+1)(1-q) - (n-k)q = k + 1 - kq - q - nq + kq = 1 + k - q - nq$.
(B) $\implies \Delta_k \geq 0$ when $k \geq nq + q - 1$
$\Delta_k < 0$ otherwise.
(C) $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
(D) $\mu(k) < \mu(k+1)$, for $k < nq$, and $\mu(k) \geq \mu(k+1)$ for $k \geq nq$.
(E) $\implies \mu(nq)$ is the largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
(F) $\mu(nq)$ larger than the average in sum.
(G) $\implies \binom{n}{k} q^k (1-q)^{n-k} \geq \frac{1}{n+1}$.
(H) $\implies \binom{n}{nq} \geq \frac{1}{n+1} q^{-nq} (1-q)^{-(n-nq)} = \frac{1}{n+1} 2^{n\mathbb{H}(q)}$. ■

23.2.3.3 Generalization...

Corollary 23.2.6. *We have:*
(i) $q \in [0, 1/2] \Rightarrow \binom{n}{\lfloor nq \rfloor} \leq 2^{n\mathbb{H}(q)}$. *(ii)* $q \in [1/2, 1]$ $\binom{n}{\lceil nq \rceil} \leq 2^{n\mathbb{H}(q)}$.

(iii) $q \in [1/2, 1] \Rightarrow \frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{\lfloor nq \rfloor}$. *(iv)* $q \in [0, 1/2] \Rightarrow \frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{\lceil nq \rceil}$.

Proof is straightforward but tedious.

23.2.3.4 What we have...

(A) Proved that $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$.
(B) Estimate is loose.
(C) Sanity check...
  (I) A sequence of n bits generated by coin with probability q for head.
  (II) By Chernoff inequality... roughly nq heads in this sequence.
  (III) Generated sequence Y belongs to $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$ possible sequences .
  (IV) ...of similar probability.
  (V) $\implies \mathbb{H}(Y) = n\mathbb{H}(q) \approx \lg \binom{n}{nq}$.

23.2.4 Extracting randomness

23.2.4.1 Just one bit...

question Given a coin C with:
p: Probability for head.
$q = 1 - p$: Probability for tail.
Q: How to get **one** true random bit, by flipping C.
Describe an algorithm!

23.2.4.2 Extracting randomness...

Entropy can be interpreted as the amount of unbiased random coin flips can be extracted from a random variable.

Definition 23.2.7. An extraction function **Ext** takes as input the value of a random variable X and outputs a sequence of bits y, such that $\mathbf{Pr}\left[\mathbf{Ext}(X) = y \,\middle|\, |y| = k\right] = \frac{1}{2^k}$, whenever $\mathbf{Pr}[|y| = k] > 0$, where $|y|$ denotes the length of y.

23.2.4.3 Extracting randomness...

(A) X: uniform random integer variable out of $0, \ldots, 7$.
(B) $\mathbf{Ext}(X)$: binary representation of x.
(C) Def. subtle: all extracted seqs of same len have same probability.
(D) Another example of extraction scheme:
  (A) X: uniform random integer variable $0, \ldots, 11$.
  (B) $\mathbf{Ext}(x)$: output the binary representation for x if $0 \leq x \leq 7$.
  (C) If x is between 8 and 11?
  (D) Idea... Output binary representation of $x - 8$ as a two bit number.
(E) A valid extractor...
  $\mathbf{Pr}\Big[\mathbf{Ext}(X) = 00 \;\Big|\; |\mathbf{Ext}(X)| = 2\Big] = \frac{1}{4}$,

23.2.4.4 Technical lemma

The following is obvious, but we provide a proof anyway.

Lemma 23.2.8. *Let x/y be a faction, such that $x/y < 1$. Then, for any i, we have $x/y < (x+i)/(y+i)$.*

Proof: We need to prove that $x(y+i) - (x+i)y < 0$. The left size is equal to $i(x-y)$, but since $y > x$ (as $x/y < 1$), this quantity is negative, as required. ■

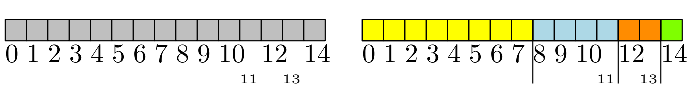
23.2.4.5 A uniform variable extractor...

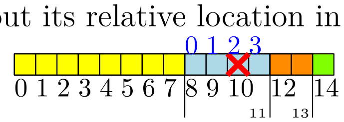
Theorem 23.2.9. *(A) X: random variable chosen uniformly at random from $\{0, \ldots, m-1\}$.*
(B) Then there is an extraction function for X:
  (A) outputs on average at least

$$\lfloor \lg m \rfloor - 1 = \lfloor \mathbb{H}(X) \rfloor - 1$$

independent and unbiased bits.

23.2.4.6 Proof

(A) m: A sum of unique powers of 2, namely $m = \sum_i a_i 2^i$, where $a_i \in \{0, 1\}$.
(B) Example: 

(C) decomposed $\{0, \ldots, m-1\}$ into disjoint union of blocks sizes are powers of 2.
(D) If x is in block 2^k, output its relative location in the block in binary representation.
(E) Example: $x = 10$: 

  then falls into block 2^2...
  x relative location is 2. Output 2 written using two bits,
  Output: "10".

23.2.4.7 Proof continued

(A) Valid extractor...
(B) Theorem holds if m is a power of two. Only one block.
(C) m not a power of 2...
(D) X falls in block of size 2^k: then output k complete random bits..
... entropy is k.
(E) Let $2^k < m < 2^{k+1}$ biggest block.
(F) $u = \left\lfloor \lg(m - 2^k) \right\rfloor < k$.
There must be a block of size u in the decomposition of m.
(G) two blocks in decomposition of m: sizes 2^k and 2^u.
(H) Largest two blocks...
(I) $2^k + 2 * 2^u > m \implies 2^{u+1} + 2^k - m > 0$.
(J) Y: random variable = number of bits output by extractor.

23.2.4.8 Proof continued

(A) By lemma, since $\frac{m-2^k}{m} < 1$:

$$\frac{m-2^k}{m} \leq \frac{m - 2^k + \left(2^{u+1} + 2^k - m\right)}{m \qquad +(2^{u+1} + 2^k - m)} = \frac{2^{u+1}}{2^{u+1} + 2^k}.$$

(B) By induction (assumed holds for all numbers smaller than m):

$$\mathbf{E}[Y] \geq \frac{2^k}{m} k + \frac{m - 2^k}{m}\left(\underbrace{\left\lfloor \lg(m - 2^k) \right\rfloor}_{u} -1\right)$$

$$= \frac{2^k}{m} k + \frac{m - 2^k}{m}(\underbrace{k - k}_{=0} + u - 1)$$

$$= k + \frac{m-2^k}{m}(u - k - 1)$$

23.2.4.9 Proof continued..

(A) We have:

$$\begin{aligned}\mathbf{E}\left[Y\right] &\geq k + \frac{m - 2^k}{m}(u - k - 1)\\ &\geq k + \frac{2^{u+1}}{2^{u+1} + 2^k}(u - k - 1)\\ &= k - \frac{2^{u+1}}{2^{u+1} + 2^k}(1 + k - u),\end{aligned}$$

since $u - k - 1 \leq 0$ as $k > u$.
(B) If $u = k - 1$, then $\mathbf{E}[Y] \geq k - \frac{1}{2} \cdot 2 = k - 1$, as required.
(C) If $u = k - 2$ then $\mathbf{E}[Y] \geq k - \frac{1}{3} \cdot 3 = k - 1$.

23.2.4.10 Proof continued.....

(A) $\mathbf{E}[Y] \geq k - \frac{2^{u+1}}{2^{u+1}+2^k}(1 + k - u)$.
And $u - k - 1 \leq 0$ as $k > u$.

(B) If $u < k - 2$ then

$$\begin{aligned}\mathbf{E}[Y] &\geq k - \frac{2^{u+1}}{2^k}(1 + k - u)\\ &= k - \frac{k - u + 1}{2^{k-u-1}}\\ &= k - \frac{2 + (k - u - 1)}{2^{k-u-1}}\\ &\geq k - 1,\end{aligned}$$

since $(2 + i)/2^i \leq 1$ for $i \geq 2$.