Network Flow

Lecture 12
October 3, 2013
Network flow

1. Transfer as much “merchandise” as possible from one point to another.
2. Wireless network, transfer a large file from $s$ to $t$.
3. Limited capacities.
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Network: Definition

1. Given a network with capacities on each connection.
2. Q: How much “flow” can transfer from source $s$ to a sink $t$?
3. The flow is *splitable*.
5. Internet is packet base, so not quite splitable.

---

**Definition**

- $G = (V, E)$: a *directed* graph.
- $\forall (u \rightarrow v) \in E(G)$: *capacity* $c(u, v) \geq 0$.
- $(u \rightarrow v) \not\in G \implies c(u, v) = 0$.
- $s$: source vertex, $t$: target sink vertex.
- $G$, $s$, $t$ and $c(\cdot)$: form *flow network* or network.
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Network Example

1. All flow from the source ends up in the sink.
2. Flow on edge: non-negative quantity $\leq$ capacity of edge.
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All flow from the source ends up in the sink.

Flow on edge: non-negative quantity \( \leq \) capacity of edge.
Flow definition

Definition (flow)

*flow* in network is a function \( f(\cdot, \cdot) : E(G) \to \mathbb{R} \):

(A) **Bounded by capacity:**
\[ \forall (u \to v) \in E \quad f(u, v) \leq c(u, v). \]

(B) **Anti symmetry:**
\[ \forall u, v \quad f(u, v) = -f(v, u). \]

(C) Two special vertices: (i) the *source* \( s \) and the *sink* \( t \).

(D) **Conservation of flow** (Kirchhoff’s Current Law):
\[ \forall u \in V \setminus \{ s, t \} \quad \sum_v f(u, v) = 0. \]

*flow/value* of \( f \):
\[ |f| = \sum_{v \in V} f(s, v). \]
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Problem: Max Flow

1. Flow on edge can be negative (i.e., positive flow on edge in other direction).

Problem (Maximum flow)

Given a network $G$ find the maximum flow in $G$. Namely, compute a legal flow $f$ such that $|f|$ is maximized.
Part II

Some properties of flows and residual networks
Flow across sets of vertices

∀X, Y ⊆ V, let \( f(X, Y) = \sum_{x \in X, y \in Y} f(x, y) \).

\( f(v, S) = f(\{v\}, S) \), where \( v \in V(G) \).

Observation

\( |f| = f(s, V) \).
Basic properties of flows: (i)

Lemma

For a flow $f$, the following properties holds:

(i) $\forall u \in V(G)$ we have $f(u, u) = 0$,

Proof.

Holds since $(u \rightarrow u)$ is not an edge in $G$.

$(u \rightarrow u)$ capacity is zero,

Flow on $(u \rightarrow u)$ is zero.
Lemma

For a flow \( f \), the following properties holds:

1. \( \forall u \in V(G) \) we have \( f(u, u) = 0 \),

Proof.

Holds since \( (u \rightarrow u) \) it not an edge in \( G \).
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For a flow $f$, the following properties holds:

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Proof.

Holds since $(u \rightarrow u)$ it not an edge in $G$. $(u \rightarrow u)$ capacity is zero,
Flow on $(u \rightarrow u)$ is zero.
Lemma

For a flow $f$, the following properties holds:

(ii) $\forall X \subseteq V$ we have $f(X, X) = 0$,

Proof.

$$f(X, X) = \sum_{\{u,v\} \subseteq X, u \neq v} (f(u, v) + f(v, u)) + \sum_{u \in X} f(u, u)$$

$$= \sum_{\{u,v\} \subseteq X, u \neq v} (f(u, v) - f(u, v)) + \sum_{u \in X} 0 = 0,$$

by the anti-symmetry property of flow.
Basic properties of flows: (iii)

Lemma

For a flow \( f \), the following properties holds:

(iii) \( \forall X, Y \subseteq V \) we have \( f(X, Y) = -f(Y, X) \).

Proof.

By the anti-symmetry of flow, as

\[
f(X, Y) = \sum_{x \in X, y \in Y} f(x, y) = -\sum_{x \in X, y \in Y} f(y, x) = -f(Y, X).
\]
Basic properties of flows: (iv)

Lemma

For a flow \( f \), the following properties holds:

(iv) \( \forall X, Y, Z \subseteq V \) such that \( X \cap Y = \emptyset \) we have that

\[
\begin{align*}
    f(X \cup Y, Z) &= f(X, Z) + f(Y, Z) \\
    f(Z, X \cup Y) &= f(Z, X) + f(Z, Y).
\end{align*}
\]

Proof.

Follows from definition. (Check!)
Lemma

For a flow $f$, the following properties hold:

$(v)$ $\forall u \in V \setminus \{s, t\}$, we have $f(u, V) = f(V, u) = 0$.

Proof.

This is a restatement of the conservation of flow property.
Basic properties of flows: summary

Lemma

For a flow $f$, the following properties holds:

(i) $\forall u \in V(G) \text{ we have } f(u, u) = 0$,
(ii) $\forall X \subseteq V \text{ we have } f(X, X) = 0$,
(iii) $\forall X, Y \subseteq V \text{ we have } f(X, Y) = -f(Y, X)$,
(iv) $\forall X, Y, Z \subseteq V \text{ such that } X \cap Y = \emptyset \text{ we have that }$
    $f(X \cup Y, Z) = f(X, Z) + f(Y, Z) \text{ and }$
    $f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$.
(v) For all $u \in V \setminus \{s, t\}$, we have $f(u, V) = f(V, u) = 0$. 
All flow gets to the sink

Claim

\[ |f| = f(V, t). \]

Proof.

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Proof.

\[ |f| = f(s, V) = f(V \setminus (V \setminus \{s\}), V) \]
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Proof.

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= f(V, V) - f(V \setminus \{s\}, V)
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Claim

\[ |f| = f(V, t). \]

Proof.

\[
|f| = f(V, V) - f(V \setminus \{s\}, V) \\
= -f(V \setminus \{s\}, V) = f(V, V \setminus \{s\})
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Claim

\[ |f| = f(V, t). \]

Proof.

\[
|f| = f(V, V) - f(V \setminus \{s\}, V) \\
= f(V, V \setminus \{s\}) \\
= f(V, t) + f(V, V \setminus \{s, t\})
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Since \( f(V, V) = 0 \) by (i) and \( f(V, u) = 0 \) by (iv).
All flow gets to the sink

Claim

$|f| = f(V, t)$. 

Proof.

\[
|f| = f(V, t) + f(V, V \setminus \{s, t\}) \\
= f(V, t) + \sum_{u \in V \setminus \{s,t\}} f(V, u) \\
= f(V, t) + \sum_{u \in V \setminus \{s,t\}} 0 \\
= f(V, t),
\]

Since $f(V, V) = 0$ by (i) and $f(V, u) = 0$ by (iv).
Residual capacity

Definition

c: capacity, f: flow.

The residual capacity of an edge \((u \to v)\) is

\[ c_f(u, v) = c(u, v) - f(u, v). \]

1. residual capacity \(c_f(u, v)\) on \((u \to v)\) = amount of unused capacity on \((u \to v)\).

2. ... next construct graph with all edges not being fully used by \(f\).
Residual capacity

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Graph Residual graph

\[ f(u, w) = -f(w, u) = -1 \implies c_f(u, w) = 10 - (-1) = 11. \]
Graph

Residual graph

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**Residual graph: Definition**

**Definition**

Given $f$, $G = (V, E)$ and $c$, as above, the **residual graph** (or **residual network**) of $G$ and $f$ is the graph $G_f = (V, E_f)$ where

$$E_f = \{(u, v) \in V \times V \mid c_f(u, v) > 0\}.$$

1. $(u \to v) \in E$: might induce two edges in $E_f$
2. If $(u \to v) \in E$, $f(u, v) < c(u, v)$ and $(v \to u) \notin E(G)$
3. $c_f(u, v) = c(u, v) - f(u, v) > 0$
4. ... and $(u \to v) \in E_f$. Also,

$$c_f(v, u) = c(v, u) - f(v, u) = 0 - (-f(u, v)) = f(u, v),$$

since $c(v, u) = 0$ as $(v \to u)$ is not an edge of $G$.
5. $(v \to u) \in E_f$. 
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Residual network properties

Since every edge of $G$ induces at most two edges in $G_f$, it follows that $G_f$ has at most twice the number of edges of $G$; formally, $|E_f| \leq 2 |E|$.

Lemma

Given a flow $f$ defined over a network $G$, then the residual network $G_f$ together with $c_f$ form a flow network.

Proof.

One need to verify that $c_f(\cdot)$ is always a non-negative function, which is true by the definition of $E_f$. 
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**Proof.**

One need to verify that $c_f(\cdot)$ is always a non-negative function, which is true by the definition of $E_f$. 

$\square$
Increasing the flow

**Lemma**

\( G(V, E) \), a flow \( f \), and \( h \) a flow in \( G_f \). \( G_f \): residual network of \( f \).
Then \( f + h \) is a flow in \( G \) and its capacity is \( |f + h| = |f| + |h| \).

**proof**

By definition: \((f + h)(u, v) = f(u, v) + h(u, v)\) and thus \((f + h)(X, Y) = f(X, Y) + h(X, Y)\). Verify legal...

1. Anti symmetry:
   \((f + h)(u, v) = f(u, v) + h(u, v) =
   -f(v, u) - h(v, u) = -(f + h)(v, u)\).

2. Bounded by capacity:

\[
(f + h)(u, v) \leq f(u, v) + h(u, v) \leq f(u, v) + c_f(u, v) \\
= f(u, v) + (c(u, v) - f(u, v)) = c(u, v).
\]
In increasing the flow

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Then \(f + h\) is a flow in \(G\) and its capacity is \(|f + h| = |f| + |h|\).

**proof**

By definition: \((f + h)(u, v) = f(u, v) + h(u, v)\) and thus \((f + h)(X, Y) = f(X, Y) + h(X, Y)\). Verify legal...

1. Anti symmetry: \((f + h)(u, v) = f(u, v) + h(u, v) = -f(v, u) - h(v, u) = -(f + h)(v, u)\).

2. Bounded by capacity:

\[
(f + h)(u, v) \leq f(u, v) + h(u, v) \leq f(u, v) + c_f(u, v) = f(u, v) + (c(u, v) - f(u, v)) = c(u, v).
\]
Increasing the flow

**Lemma**

\( G(V, E) \), a flow \( f \), and \( h \) a flow in \( G_f \). \( G_f \): residual network of \( f \).

Then \( f + h \) is a flow in \( G \) and its capacity is \( |f + h| = |f| + |h| \).

**proof**

By definition: \( (f + h)(u, v) = f(u, v) + h(u, v) \) and thus \( (f + h)(X, Y) = f(X, Y) + h(X, Y) \).

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For $u \in V - s - t$ we have

$$(f + h)(u, V) = f(u, V) + h(u, V) = 0 + 0 = 0$$

and as such $f + h$ comply with the conservation of flow requirement.

Total flow is

$$|f + h| = (f + h)(s, V) = f(s, V) + h(s, V) = |f| + |h|.$$
Increasing the flow – proof continued

Proof continued

1. For \( u \in V - s - t \) we have
\[
(f + h)(u, V) = f(u, V) + h(u, V) = 0 + 0 = 0
\]
and as such \( f + h \) comply with the conservation of flow requirement.

2. Total flow is
\[
|f + h| = (f + h)(s, V) = f(s, V) + h(s, V) = |f| + |h|.
\]
Augmenting path

**Definition**

For $G$ and a flow $f$, a path $\pi$ in $G_f$ between $s$ and $t$ is an **augmenting path**.
More on augmenting paths

1. $\pi$: augmenting path.
2. All edges of $\pi$ have positive capacity in $G_f$.
3. ... otherwise not in $E_f$.
4. $f$, $\pi$: can improve $f$ by pushing positive flow along $\pi$. 
More on augmenting paths

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3. ... otherwise not in \( E_f \).
4. \( f, \pi \): can improve \( f \) by pushing positive flow along \( \pi \).
Residual capacity

**Definition**

\(\pi\): augmenting path of \(f\).

\(c_f(\pi)\): maximum amount of flow can push on \(\pi\).

\(c_f(\pi)\) is *residual capacity* of \(\pi\).

Formally,

\[
c_f(\pi) = \min_{(u \rightarrow v) \in \pi} c_f(u, v).
\]
An example of an augmenting path

(A) Flow

(B) Residual network

(C) Augmenting path

(D) New flow
Flow along augmenting path

\[ f_\pi(u, v) = \begin{cases} 
  c_f(\pi) & \text{if } (u \rightarrow v) \text{ is in } \pi \\
  -c_f(\pi) & \text{if } (v \rightarrow u) \text{ is in } \pi \\
  0 & \text{otherwise.}
\end{cases} \]
Increase flow by augmenting flow

**Lemma**

\( \pi : \text{augmenting path}. \ f_\pi \text{ is flow in } G_f \text{ and } |f_\pi| = c_f(\pi) > 0. \)

Get bigger flow...

**Lemma**

Let \( f \) be a flow, and let \( \pi \) be an augmenting path for \( f \). Then \( f + f_\pi \) is a “better” flow. Namely, \( |f + f_\pi| = |f| + |f_\pi| > |f| \).
Increase flow by augmenting flow

**Lemma**

\( \pi \): augmenting path. \( f_\pi \) is flow in \( G_f \) and \(|f_\pi| = c_f(\pi) > 0 \).

Get bigger flow...

**Lemma**

Let \( f \) be a flow, and let \( \pi \) be an augmenting path for \( f \). Then \( f + f_\pi \) is a “better” flow. Namely, \(|f + f_\pi| = |f| + |f_\pi| > |f| \).
1. Namely, $f + f_\pi$ is flow with larger value than $f$.

2. Can this flow be improved?

3. $s$ is disconnected from $t$ in this residual network.

4. unable to push more flow.

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6. Is that a global maximum?

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Flowing into the wall

1. Namely, $f + f_\pi$ is flow with larger value than $f$.

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The Ford-Fulkerson method

\[
\text{algFordFulkerson}(G, c) \\
\begin{align*}
\text{begin} & \\
\quad f & \leftarrow \text{Zero flow on } G \\
\quad \text{while } (G_f \text{ has augmenting path } p) \text{ do} & \\
\quad \quad (* \text{ Recompute } G_f \text{ for this check } *) & \\
\quad \quad f & \leftarrow f + f_p \\
\text{return } f & \\
\text{end}
\end{align*}
\]
Part III

On maximum flows
Some definitions

Definition

\((S, T)\): directed cut in flow network \(G = (V, E)\).
A partition of \(V\) into \(S\) and \(T = V \setminus S\), such that \(s \in S\) and \(t \in T\).
Some definitions

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Definition

The net flow of \(f\) across a cut \((S, T)\) is
\[f(S, T) = \sum_{s \in S, t \in T} f(s, t)\.]
Some definitions

**Definition**

$(S, T)$: *directed cut* in flow network $G = (V, E)$. A partition of $V$ into $S$ and $T = V \setminus S$, such that $s \in S$ and $t \in T$.

**Definition**

The net *flow of $f$ across a cut* $(S, T)$ is $f(S, T) = \sum_{s \in S, t \in T} f(s, t)$.

**Definition**

The *capacity* of $(S, T)$ is $c(S, T) = \sum_{s \in S, t \in T} c(s, t)$.
Some definitions

Definition

\((S, T)\): **directed cut** in flow network \(G = (V, E)\).
A partition of \(V\) into \(S\) and \(T = V \setminus S\), such that \(s \in S\) and \(t \in T\).

Definition

The net **flow of \(f\) across a cut** \((S, T)\) is
\[
f(S, T) = \sum_{s \in S, t \in T} f(s, t).
\]

Definition

The **capacity** of \((S, T)\) is \(c(S, T) = \sum_{s \in S, t \in T} c(s, t)\).

Definition

The **minimum cut** is the cut in \(G\) with the minimum capacity.
Flow across cut is the whole flow

**Lemma**

\[ G, f, s, t. \quad (S, T): \text{cut of } G. \]

Then \( f(S, T) = |f| \).

**Proof.**

\[
\begin{align*}
f(S, T) &= f(S, V) - f(S, S) = f(S, V) \\
&= f(s, V) + f(S - s, V) = f(s, V) \\
&= |f|
\end{align*}
\]

since \( T = V \setminus S \), and \( f(S - s, V) = \sum_{u \in S - s} f(u, V) = 0 \) (note that \( u \) can not be \( t \) as \( t \in T \)).
The flow in a network is upper bounded by the capacity of any cut \((S, T)\) in \(G\).

Proof.

Consider a cut \((S, T)\). We have
\[
|f| = f(S, T) = \sum_{u \in S, v \in T} f(u, v) \leq \sum_{u \in S, v \in T} c(u, v) = c(S, T).
\]
Key observation

Maximum flow is bounded by the capacity of the minimum cut.

Surprisingly...

Maximum flow is exactly the value of the minimum cut.
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Maximum flow is bounded by the capacity of the minimum cut.

Surprisingly...
Maximum flow is exactly the value of the minimum cut.
The Min-Cut Max-Flow Theorem

Theorem (Max-flow min-cut theorem)

If \( f \) is a flow in a flow network \( G = (V, E) \) with source \( s \) and sink \( t \), then the following conditions are equivalent:

(A) \( f \) is a maximum flow in \( G \).

(B) The residual network \( G_f \) contains no augmenting paths.

(C) \(|f| = c(S, T)\) for some cut \((S, T)\) of \( G \). And \((S, T)\) is a minimum cut in \( G \).
Proof: \((A) \implies (B)\):

Proof.

\((A) \implies (B)\): By contradiction. If there was an augmenting path \(p\) then \(c_f(p) > 0\), and we can generate a new flow \(f + f_p\), such that \(|f + f_p| = |f| + c_f(p) > |f|\). A contradiction as \(f\) is a maximum flow.
Proof: \((B) \Rightarrow (C):\)

Proof.

\(s\) and \(t\) are disconnected in \(G_f\).

Set

\[ S = \{ v \mid \text{Exists a path between } s \text{ and } v \text{ in } G_f \} \quad \text{and} \quad T = V \setminus S. \]

Have: \(s \in S, \ t \in T, \ \forall u \in S \text{ and } \forall v \in T: \ f(u, v) = c(u, v). \)

By contradiction: \(\exists u \in S, \ v \in T\) s.t. \(f(u, v) < c(u, v) \implies (u \rightarrow v) \in E_f \implies v\) would be reachable from \(s\) in \(G_f\).

Contradiction.

\[ \implies |f| = f(S, T) = c(S, T). \]

\((S, T)\) must be mincut. Otherwise \(\exists (S', T'):\)

\[ c(S', T') < c(S, T) = f(S, T) = |f|, \]

But... \( |f| = f(S', T') \leq c(S', T') \). A contradiction.
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Contradiction.

\[
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**Proof.**

$s$ and $t$ are disconnected in $G_f$.

Set

\[
S = \{ v \mid \text{Exists a path between } s \text{ and } v \text{ in } G_f \}, \quad T = V \setminus S.
\]

Have: $s \in S$, $t \in T$, $\forall u \in S$ and $\forall v \in T$: $f(u, v) = c(u, v)$.

By contradiction: $\exists u \in S$, $v \in T$ s.t. $f(u, v) < c(u, v) \implies (u \rightarrow v) \in E_f \implies v$ would be reachable from $s$ in $G_f$.

Contradiction.

$\implies |f| = f(S, T) = c(S, T)$.

\((S, T)\) must be mincut. Otherwise $\exists (S', T')$:

\[c(S', T') < c(S, T) = f(S, T') = |f|,\]

But... $|f| = f(S', T') \leq c(S', T')$. A contradiction.
Proof: \((C) \implies (A)\):

\begin{proof}
Well, for any cut \((U, V)\), we know that \(|f| \leq c(U, V)\). This implies that if \(|f| = c(S, T)\) then the flow can not be any larger, and it is thus a maximum flow.
\end{proof}
Implications

1. The max-flow min-cut theorem \(\implies\) if \texttt{algFordFulkerson} terminates, then computed max flow.

2. Does not imply \texttt{algFordFulkerson} always terminates.

3. \texttt{algFordFulkerson} might not terminate.
Part IV

Non-termination of Ford-Fulkerson
Ford-Fulkerson runs in vain

$M$: large positive integer.

$\alpha = (\sqrt{5} - 1)/2 \approx 0.618$.

$\alpha < 1$,

$1 - \alpha < \alpha$.

Maximum flow in this network is: $2M + 1$. 

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\begin{figure}
\centering
\includegraphics[width=\textwidth]{network_diagram}
\end{figure}
Some algebra...

For $\alpha = \frac{\sqrt{5} - 1}{2}$:

$\alpha^2$
Some algebra...

For \( \alpha = \frac{\sqrt{5} - 1}{2} \):

\[
\alpha^2 = \left( \frac{\sqrt{5} - 1}{2} \right)^2
\]
Some algebra...

For \( \alpha = \frac{\sqrt{5} - 1}{2} \):

\[
\alpha^2 = \left( \frac{\sqrt{5} - 1}{2} \right)^2 = \frac{1}{4} \left( \sqrt{5} - 1 \right)^2
\]
Some algebra...

For $\alpha = \frac{\sqrt{5} - 1}{2}$:

$$\alpha^2 = \left( \frac{\sqrt{5} - 1}{2} \right)^2 = \frac{1}{4} (\sqrt{5} - 1)^2 = \frac{1}{4} (5 - 2\sqrt{5} + 1)$$
Some algebra...

For $\alpha = \frac{\sqrt{5} - 1}{2}$:

$$\alpha^2 = \left(\frac{\sqrt{5} - 1}{2}\right)^2 = \frac{1}{4} (\sqrt{5} - 1)^2 = \frac{1}{4} (5 - 2\sqrt{5} + 1)$$

$$= 1 + \frac{1}{4} (2 - 2\sqrt{5})$$
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Some algebra...

For $\alpha = \frac{\sqrt{5} - 1}{2}$:

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$$= 1 + \frac{1}{2} (1 - \sqrt{5})$$

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For \( \alpha = \frac{\sqrt{5} - 1}{2} \):

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\]

\[
= 1 + \frac{1}{4} (2 - 2\sqrt{5})
\]

\[
= 1 + \frac{1}{2} (1 - \sqrt{5})
\]

\[
= 1 - \frac{\sqrt{5} - 1}{2}
\]

\[
= 1 - \alpha.
\]
Some algebra...

**Claim**

Given: \( \alpha = (\sqrt{5} - 1)/2 \) and \( \alpha^2 = 1 - \alpha \).

\[ \implies \forall i \quad \alpha^i - \alpha^{i+1} = \alpha^{i+2} \]

**Proof.**

\[ \alpha^i - \alpha^{i+1} = \alpha^i(1 - \alpha) = \alpha^i\alpha^2 = \alpha^{i+2}. \]
The network
# | Augment. path $\pi$ | $C_\pi$ | New residual network

0. | ![Augment. path 0](image0) | | |

1. | ![Augment. path 1](image1) | | |
<table>
<thead>
<tr>
<th>#</th>
<th>Augment. path $\pi$</th>
<th>$c_\pi$</th>
<th>New residual network</th>
</tr>
</thead>
<tbody>
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<td>0.</td>
<td><img src="#" alt="Graph" /></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1.</td>
<td><img src="#" alt="Graph" /></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Let it flow...

<table>
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<tbody>
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<td>1</td>
<td><img src="graphs/0_sol.png" alt="Graph" /></td>
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<td><img src="graphs/1_sol.png" alt="Graph" /></td>
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<tr>
<td>#</td>
<td>Augment. path $\pi$</td>
<td>$c_\pi$</td>
<td>New residual network</td>
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<tr>
<td>0.</td>
<td><img src="image0.png" alt="Augment. path diagram" /></td>
<td>1</td>
<td><img src="image1.png" alt="New residual network diagram" /></td>
</tr>
<tr>
<td>1.</td>
<td><img src="image2.png" alt="Augment. path diagram" /></td>
<td>$\alpha$</td>
<td><img src="image3.png" alt="New residual network diagram" /></td>
</tr>
<tr>
<td>#</td>
<td>Augment. path $\pi$</td>
<td>$c_\pi$</td>
<td>New residual network</td>
</tr>
<tr>
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</tr>
<tr>
<td>0.</td>
<td><img src="image" alt="Diagram 0" /></td>
<td>1</td>
<td><img src="image" alt="Diagram 1" /></td>
</tr>
<tr>
<td>1.</td>
<td><img src="image" alt="Diagram 1" /></td>
<td>$\alpha$</td>
<td><img src="image" alt="Diagram 1" /></td>
</tr>
</tbody>
</table>
Let it flow II

<table>
<thead>
<tr>
<th>#</th>
<th>Augment. path $\pi$</th>
<th>$c_\pi$</th>
<th>New residual network</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td><img src="image1" alt="Augment. path $\pi_1$" /></td>
<td>$\alpha$</td>
<td><img src="image2" alt="New residual network" /></td>
</tr>
<tr>
<td>2.</td>
<td><img src="image3" alt="Augment. path $\pi_2$" /></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>#</th>
<th>Augment. path $\pi$</th>
<th>$c_\pi$</th>
<th>New residual network</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td><img src="image1.png" alt="Diagram 1" /></td>
<td>$\alpha$</td>
<td><img src="image2.png" alt="Diagram 2" /></td>
</tr>
<tr>
<td>2.</td>
<td><img src="image3.png" alt="Diagram 3" /></td>
<td>$\alpha$</td>
<td><img src="image4.png" alt="Diagram 4" /></td>
</tr>
<tr>
<td>#</td>
<td>Augment. path $\pi$</td>
<td>$c_\pi$</td>
<td>New residual network</td>
</tr>
<tr>
<td>----</td>
<td>---------------------</td>
<td>---------</td>
<td>---------------------</td>
</tr>
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<td>1.</td>
<td><img src="image1.png" alt="Diagram 1" /></td>
<td>$\alpha$</td>
<td><img src="image2.png" alt="Diagram 2" /></td>
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<td>2.</td>
<td><img src="image3.png" alt="Diagram 3" /></td>
<td>$\alpha$</td>
<td><img src="image4.png" alt="Diagram 4" /></td>
</tr>
</tbody>
</table>
Let it flow II

2. \[ \alpha^2 \]

\[ w \xrightarrow{1} z \xrightarrow{1} y \xrightarrow{\alpha} x \]

\[ p_2 \]

\[ w \xrightarrow{\alpha^2} z \xrightarrow{1 - \alpha^2} y \xrightarrow{\alpha} x \]

3. \[ \alpha^2 \]

\[ w \xrightarrow{1} z \xrightarrow{1} y \xrightarrow{\alpha} x \]

\[ p_1 \]

\[ w \xrightarrow{1 - \alpha^2} z \xrightarrow{\alpha^2} y \xrightarrow{\alpha^2} x \]

\[ \alpha - \alpha^2 = \alpha^3 \]
3. Let it flow

\[ z \rightarrow y \rightarrow x \]

\[ \alpha \]

\[ w \rightarrow t \]

\[ p_1 \]

\[ \alpha^2 \]

\[ 1 \]

\[ \alpha^2 \]

\[ 1 - \alpha^2 \]

\[ \alpha^3 \]

4. Let it flow

\[ z \rightarrow y \rightarrow x \]

\[ \alpha \]

\[ w \rightarrow t \]

\[ p_3 \]

\[ \alpha^2 \]

\[ 1 \]

\[ \alpha^3 \]
Let it flow III

<table>
<thead>
<tr>
<th>moves</th>
<th>Residual network after</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$w \rightarrow 1 \rightarrow z \rightarrow 1 \rightarrow y \rightarrow \alpha \rightarrow x$</td>
</tr>
<tr>
<td>moves 0, (1, 2, 3, 4)</td>
<td>$w \rightarrow \alpha^2 \rightarrow z \rightarrow 1 \rightarrow y \rightarrow \alpha^3 \rightarrow x$</td>
</tr>
<tr>
<td>moves 0, (1, 2, 3, 4)$^2$</td>
<td>$w \rightarrow \alpha^4 \rightarrow z \rightarrow 1 \rightarrow y \rightarrow \alpha^5 \rightarrow x$</td>
</tr>
<tr>
<td>0.$(1, 2, 3, 4)^i$</td>
<td>$w \rightarrow \alpha^{2i} \rightarrow z \rightarrow 1 \rightarrow y \rightarrow \alpha^{2i+1} \rightarrow x$</td>
</tr>
</tbody>
</table>

Namely, the algorithm never terminates.