Network Flow

Transfer as much “merchandise” as possible from one point to another.
Wireless network, transfer a large file from $s$ to $t$.
Limited capacities.

Network: Definition

- Given a network with capacities on each connection.
- Q: How much “flow” can transfer from source $s$ to a sink $t$?
- The flow is *splitable*.
- Network examples: water pipes moving water. Electricity network.
- Internet is packet base, so not quite splitable.

**Definition**

- $G = (V, E)$: a *directed* graph.
- $\forall (u \rightarrow v) \in E(G)$: *capacity* $c(u, v) \geq 0$.
- $(u \rightarrow v) \notin G \implies c(u, v) = 0$.
- $s$: *source* vertex, $t$: target *sink* vertex.
- $G$, $s$, $t$ and $c(\cdot)$: form *flow network* or *network*.
Network Example

- All flow from the source ends up in the sink.
- Flow on edge: non-negative quantity ≤ capacity of edge.

Problem: Max Flow

- Flow on edge can be negative (i.e., positive flow on edge in other direction).

Problem (Maximum flow)

Given a network $G$ find the maximum flow in $G$. Namely, compute a legal flow $f$ such that $|f|$ is maximized.

Flow definition

**Definition (flow)**

A flow in network is a function $f(\cdot, \cdot) : E(G) \rightarrow \mathbb{R}$:

- **Bounded by capacity**: \( \forall (u \rightarrow v) \in E \quad f(u, v) \leq c(u, v) \).
- **Anti symmetry**: \( \forall u, v \quad f(u, v) = -f(v, u) \).
- **Conservation of flow** (Kirchhoff’s Current Law): \( \forall u \in V \setminus \{s, t\} \quad \sum_v f(u, v) = 0 \).

Flow / value of $f$: \( |f| = \sum_{v \in V} f(s, v) \).

Part II

Some properties of flows and residual networks
Flow across sets of vertices

∀ X, Y ⊆ V, let \( f(X, Y) = \sum_{x \in X, y \in Y} f(x, y) \).

\( f(v, S) = f(\{v\}, S) \), where \( v \in V(G) \).

Observation

\(|f| = f(s, V)|.

Basic properties of flows: (i)

Lemma

For a flow \( f \), the following properties holds:

(i) \( \forall u \in V(G) \) we have \( f(u, u) = 0 \).

Proof.

Holds since \((u \rightarrow u)\) it not an edge in \( G \). \( (u \rightarrow u) \) capacity is zero,
Flow on \((u \rightarrow u)\) is zero.

Basic properties of flows: (ii)

Lemma

For a flow \( f \), the following properties holds:

(ii) \( \forall X \subseteq V \) we have \( f(X, X) = 0 \).

Proof.

\[
\begin{align*}
\sum_{\{u,v\} \subseteq X, u \neq v} (f(u,v) + f(v,u)) + \sum_{u \in X} f(u,u) \\
= \sum_{\{u,v\} \subseteq X, u \neq v} (f(u,v) - f(u,v)) + \sum_{u \in X} 0 = 0,
\end{align*}
\]

by the anti-symmetry property of flow.

Basic properties of flows: (iii)

Lemma

For a flow \( f \), the following properties holds:

(iii) \( \forall X, Y \subseteq V \) we have \( f(X, Y) = -f(Y, X) \).

Proof.

By the anti-symmetry of flow, as
\[
\begin{align*}
f(X, Y) = \sum_{x \in X, y \in Y} f(x, y) = -\sum_{x \in X, y \in Y} f(y, x) = -f(Y, X).
\end{align*}
\]
Basic properties of flows: (iv)

Lemma
For a flow $f$, the following properties hold:

(iv) $\forall X, Y, Z \subseteq V$ such that $X \cap Y = \emptyset$ we have that

$$f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$$

and

$$f(Z, X \cup Y) = f(Z, X) + f(Z, Y).$$

Proof.
Follows from definition. (Check!)

Basic properties of flows: (v)

Lemma
For a flow $f$, the following properties hold:

(v) $\forall u \in V \setminus \{s, t\}$, we have $f(u, V) = f(V, u) = 0$.

Proof.
This is a restatement of the conservation of flow property.

Basic properties of flows: summary

Lemma
For a flow $f$, the following properties holds:

(i) $\forall u \in V(G)$ we have $f(u, u) = 0$.

(ii) $\forall X \subseteq V$ we have $f(X, X) = 0$.

(iii) $\forall X, Y \subseteq V$ we have $f(X, Y) = -f(Y, X)$.

(iv) $\forall X, Y, Z \subseteq V$ such that $X \cap Y = \emptyset$ we have that

$$f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$$

and

$$f(Z, X \cup Y) = f(Z, X) + f(Z, Y).$$

(v) For all $u \in V \setminus \{s, t\}$, we have $f(u, V) = f(V, u) = 0$.

All flow gets to the sink

Claim
$|f| = f(V, t)$.

Proof.
$$|f| = f(s, V) = f\left(V \setminus (V \setminus \{s\}), V\right)$$
$$= f(V, V) - f(V \setminus \{s\}, V)$$
$$= -f(V \setminus \{s\}, V)$$
$$= f(V, t) + f(V, V \setminus \{s, t\})$$
$$= f(V, t) + \sum_{u \in V \setminus \{s, t\}} f(V, u)$$
$$= f(V, t) + \sum_{u \in V \setminus \{s, t\}} 0$$
$$= f(V, t).$$

Since $f(V, V) = 0$ by (i) and $f(V, u) = 0$ by (iv).
Residual capacity

**Definition**

c: capacity, f: flow.

The **residual capacity** of an edge \((u \to v)\) is

\[
c_f(u, v) = c(u, v) - f(u, v).
\]

- residual capacity \(c_f(u, v)\) on \((u \to v)\) = amount of unused capacity on \((u \to v)\).
- ... next construct graph with all edges not being fully used by \(f\).

Residual graph: Definition

**Definition**

Given \(f, G = (V, E)\) and \(c\), as above, the **residual graph** (or **residual network**) of \(G\) and \(f\) is the graph \(G_f = (V, E_f)\) where

\[
E_f = \{ (u, v) \in V \times V \mid c_f(u, v) > 0 \}.
\]

- \((u \to v) \in E\): might induce two edges in \(E_f\)
- If \((u \to v) \in E, f(u, v) < c(u, v)\) and \((v \to u) \notin E(G)\)
- \(\implies c_f(u, v) = c(u, v) - f(u, v) > 0\)
- \(\implies (u \to v) \in E_f\). Also, \(c_f(v, u) = c(v, u) - f(v, u) = 0 - (-f(u, v)) = f(u, v)\), since \(c(v, u) = 0\) as \((v \to u)\) is not an edge of \(G\).
- \(\implies (v \to u) \in E_f\).

Residual network properties

Since every edge of \(G\) induces at most two edges in \(G_f\), it follows that \(G_f\) has at most twice the number of edges of \(G\); formally, |\(E_f\)| \(\leq 2 |E|\).

**Lemma**

Given a flow \(f\) defined over a network \(G\), then the residual network \(G_f\) together with \(c_f\) form a flow network.

**Proof.**

One need to verify that \(c_f(\cdot)\) is always a non-negative function, which is true by the definition of \(E_f\).
Increasing the flow

Lemma

\( G(V, E) \), a flow \( f \), and \( h \) a flow in \( G_f \). \( G_f \): residual network of \( f \).
Then \( f + h \) is a flow in \( G \) and its capacity is \( |f + h| = |f| + |h| \).

proof

By definition: \((f + h)(u, v) = f(u, v) + h(u, v)\) and thus \((f + h)(X, Y) = f(X, Y) + h(X, Y)\). Verify legal...

1. Anti symmetry: \((f + h)(u, v) = f(u, v) + h(u, v) = -f(v, u) - h(v, u) = -(f + h)(v, u)\).
2. Bounded by capacity:
   \[
   (f + h)(u, v) \leq f(u, v) + h(u, v) \leq f(u, v) + c_f(u, v) 
   = f(u, v) + (c(u, v) - f(u, v)) = c(u, v).
   \]

Increasing the flow – proof continued

proof continued

1. For \( u \in V - s - t \) we have \((f + h)(u, V) = f(u, V) + h(u, V) = 0 + 0 = 0\) and as such \( f + h \) comply with the conservation of flow requirement.
2. Total flow is
   \[ |f + h| = (f + h)(s, V) = f(s, V) + h(s, V) = |f| + |h| \]

More on augmenting paths

1. \( \pi \): augmenting path.
2. All edges of \( \pi \) have positive capacity in \( G_f \).
3. ... otherwise not in \( E_f \).
4. \( f, \pi \): can improve \( f \) by pushing positive flow along \( \pi \).
Residual capacity

Definition
\( \pi \): augmenting path of \( f \).
\( c_f(\pi) \): maximum amount of flow can push on \( \pi \).
\( c_f(\pi) \) is residual capacity of \( \pi \).
Formally,
\[
c_f(\pi) = \min_{(u \rightarrow v) \in \pi} c_f(u, v).
\]

Flow along augmenting path
\[
f_\pi(u, v) = \begin{cases} 
c_f(\pi) & \text{if } (u \rightarrow v) \text{ is in } \pi \\
-c_f(\pi) & \text{if } (v \rightarrow u) \text{ is in } \pi \\
0 & \text{otherwise}
\end{cases}
\]

Increase flow by augmenting flow

Lemma
\( \pi \): augmenting path. \( f_\pi \) is flow in \( G_f \) and \( |f_\pi| = c_f(\pi) > 0 \).
Get bigger flow...

Lemma
Let \( f \) be a flow, and let \( \pi \) be an augmenting path for \( f \). Then \( f + f_\pi \) is a “better” flow. Namely, \( |f + f_\pi| = |f| + |f_\pi| > |f| \).
Flowing into the wall

1. Namely, \( f + f_\pi \) is flow with larger value than \( f \).
2. Can this flow be improved? Consider residual flow...

\[ \begin{array}{c}
\text{Flowing into the wall} \\
\text{1. Namely, } f + f_\pi \text{ is flow with larger value than } f. \\
\text{2. Can this flow be improved? Consider residual flow...}
\end{array} \]

Part III

On maximum flows

The Ford-Fulkerson method

\[ \text{algFordFulkerson}(G, c) \]
\[ \begin{array}{l}
\text{begin} \\
\quad f \leftarrow \text{Zero flow on } G \\
\quad \text{while } (G_f \text{ has augmenting path } p) \text{ do} \\
\quad \quad (* \text{ Recompute } G_f \text{ for this check } *) \\
\quad \quad f \leftarrow f + f_p \\
\quad \quad \text{return } f \\
\text{end}
\end{array} \]

Some definitions

Definition

\((S, T)\): directed cut in flow network \(G = (V, E)\). A partition of \(V\) into \(S\) and \(T = V \setminus S\), such that \(s \in S\) and \(t \in T\).

Definition

The net flow of \(f\) across a cut \((S, T)\) is

\[ f(S, T) = \sum_{s \in S, t \in T} f(s, t). \]

Definition

The capacity of \((S, T)\) is

\[ c(S, T) = \sum_{s \in S, t \in T} c(s, t). \]

Definition

The minimum cut is the cut in \(G\) with the minimum capacity.
Flow across cut is the whole flow

Lemma

\( G, f, s, t \): cut of \( G \).

Then \( f(S, T) = |f| \).

Proof.

\[
f(S, T) = f(S, V) - f(S, S) = f(S, V)
  = f(s, V) + f(S - s, V) = f(s, V)
  = |f|,
\]

since \( T = V \setminus S \), and \( f(S - s, V) = \sum_{u \in S - s} f(u, V) = 0 \) (note that \( u \) cannot be \( t \) as \( t \in T \)).

Flow bounded by cut capacity

Claim

The flow in a network is upper bounded by the capacity of any cut \((S, T)\) in \( G \).

Proof.

Consider a cut \((S, T)\). We have \( |f| = f(S, T) = \sum_{u \in S, v \in T} f(u, v) \leq \sum_{u \in S, v \in T} c(u, v) = c(S, T) \).

THE POINT

Key observation

Maximum flow is bounded by the capacity of the minimum cut.

Surprisingly...

Maximum flow is exactly the value of the minimum cut.

The Min-Cut Max-Flow Theorem

Theorem (Max-flow min-cut theorem)

If \( f \) is a flow in a flow network \( G = (V, E) \) with source \( s \) and sink \( t \), then the following conditions are equivalent:

(A) \( f \) is a maximum flow in \( G \).

(B) The residual network \( G_f \) contains no augmenting paths.

(C) \( |f| = c(S, T) \) for some cut \((S, T)\) of \( G \). And \((S, T)\) is a minimum cut in \( G \).
Proof: (A) ⇒ (B):

Proof. (A) ⇒ (B): By contradiction. If there was an augmenting path \( p \) then \( c_f(p) > 0 \), and we can generate a new flow \( f + f_p \), such that \( |f + f_p| = |f| + c_f(p) > |f| \). A contradiction as \( f \) is a maximum flow.

Proof: (B) ⇒ (C):

Proof. \( s \) and \( t \) are disconnected in \( G_f \). Set \( S = \{ v \mid \text{Exists a path between } s \text{ and } v \text{ in } G_f \} \). Then \( T = V \setminus S \).

Have: \( s \in S \), \( t \in T \), \( \forall u \in S \) and \( \forall v \in T \): \( f(u, v) = c(u, v) \).

By contradiction: \( \exists u \in S, v \in T \) s.t. \( f(u, v) < c(u, v) \) ⇒ \((u \rightarrow v) \in E_f \) \( \Rightarrow v \) would be reachable from \( s \) in \( G_f \).

Contradiction.

\( \Rightarrow |f| = f(S, T) = c(S, T) \).

\( (S, T) \) must be mincut. Otherwise \( \exists (S', T') \):

\( c(S', T') < c(S, T) = f(S, T) = |f| \).

But... \( |f| = f(S', T') \leq c(S', T') \). A contradiction.

Proof: (C) ⇒ (A):

Proof. Well, for any cut \((U, V)\), we know that \( |f| \leq c(U, V) \). This implies that if \( |f| = c(S, T) \) then the flow can not be any larger, and it is thus a maximum flow.

Implications

- The max-flow min-cut theorem \( \Rightarrow \) if \textit{algFordFulkerson} terminates, then computed max flow.
- Does not imply \textit{algFordFulkerson} always terminates.
- \textit{algFordFulkerson} might not terminate.
Part IV
Non-termination of Ford-Fulkerson

Some algebra...

For $\alpha = \frac{\sqrt{5} - 1}{2}$:

$$\alpha^2 = \left(\frac{\sqrt{5} - 1}{2}\right)^2 = \frac{1}{4} (\sqrt{5} - 1)^2 = \frac{1}{4} (5 - 2\sqrt{5} + 1)$$

$$= 1 + \frac{1}{4} (2 - 2\sqrt{5})$$

$$= 1 + \frac{1}{2} (1 - \sqrt{5})$$

$$= 1 - \frac{\sqrt{5} - 1}{2}$$

$$= 1 - \alpha.$$
The network

Let it flow...

<table>
<thead>
<tr>
<th>#</th>
<th>Augment. path $\pi$</th>
<th>$c_\pi$</th>
<th>New residual network</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>1</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td>1</td>
<td><img src="image2" alt="Diagram" /></td>
<td>$\alpha$</td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Let it flow II

<table>
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<th>Augment. path $\pi$</th>
<th>$c_\pi$</th>
<th>New residual network</th>
</tr>
</thead>
<tbody>
<tr>
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<td><img src="image4" alt="Diagram" /></td>
<td>$\alpha$</td>
<td><img src="image5" alt="Diagram" /></td>
</tr>
<tr>
<td>2</td>
<td><img src="image6" alt="Diagram" /></td>
<td>$\alpha^2$</td>
<td><img src="image7" alt="Diagram" /></td>
</tr>
<tr>
<td>3</td>
<td><img src="image8" alt="Diagram" /></td>
<td>$\alpha^2$</td>
<td><img src="image9" alt="Diagram" /></td>
</tr>
</tbody>
</table>
Let it flow III

3. \[ \alpha^2 \]

4. \[ \alpha^2 \]

\[
\begin{align*}
&\text{Let it flow III} \\
&\text{moves} \quad \text{Residual network after} \\
&0 \quad \begin{array}{c}
\alpha^3 \\
\alpha^2 \alpha^2 \\
\end{array} \\
&\text{moves } 0, (1, 2, 3, 4) \quad \begin{array}{c}
\alpha^4 \\
\alpha^3 \\
\end{array} \\
&\text{moves } 0, (1, 2, 3, 4)^2 \quad \begin{array}{c}
\alpha^5 \\
\alpha(1 - \alpha^4) \\
\end{array} \\
&0.(1, 2, 3, 4)^i \quad \begin{array}{c}
\alpha^{2i+1} \\
\alpha - \alpha^{2i+1} \\
\end{array}
\end{align*}
\]

Namely, the algorithm never terminates.