Chapter 11

Randomized Algorithms III – Min Cut

CS 573: Algorithms, Fall 2013
October 1, 2013

11.1 Min Cut

11.1.1 Problem Definition

11.2 Min cut

11.2.0.1 Min cut

Definition 11.2.1. cut in G: a partition of V: S and V \ S.

Edges of the cut:

\[(S, V \setminus S) = \{uv \mid u \in S, v \in V \setminus S, \text{ and } uv \in E\}\],

\(|(S, V \setminus S)| \text{ is size of the cut}\)

minimum cut / mincut: cut in graph with min size.

11.2.0.2 Some definitions

(A) conditional probability of X given Y is

\[\Pr[X = x \mid Y = y] = \frac{\Pr[(X = x) \cap (Y = y)]}{\Pr[Y = y]}\].

\[\Pr[(X = x) \cap (Y = y)] = \Pr[X = x \mid Y = y] \cdot \Pr[Y = y].\]
(B) $X, Y$ events are *independent*, if $\Pr[X = x \cap Y = y] = \Pr[X = x] \cdot \Pr[Y = y]$.

$\implies \Pr[X = x \mid Y = y] = \Pr[X = x]$.

### 11.2.0.3 Some more probability

**Lemma 11.2.2.** $\varepsilon_1, \ldots, \varepsilon_n$: $n$ events (not necessarily independent). Then,

$$\Pr[\bigcap_{i=1}^n \varepsilon_i] = \Pr[\varepsilon_1] \ast \Pr[\varepsilon_2 \mid \varepsilon_1] \ast \Pr[\varepsilon_3 \mid \varepsilon_1 \cap \varepsilon_2] \ast \ldots \ast \Pr[\varepsilon_n \mid \varepsilon_1 \cap \ldots \cap \varepsilon_{n-1}].$$

### 11.3 The Algorithm

#### 11.3.0.4 Edge contraction...

![Graph G](image1) \hspace{1cm} ![Graph G/xy](image2)

- **edge contraction**: $e = xy$ in $G$.
- (B) ... merge $x, y$ into a single vertex.
- (C) ... remove self loops.
- (D) ... parallel edges – *multi-graph*.
- (E) ... weights/ multiplicities on the edges.

#### 11.3.0.5 Min cut in weighted graph

![Weighted Graph](image3) \hspace{1cm} ![Weighted Graph with Cut](image4)

Edge contraction implemented in $O(n)$ time:
- (A) Graph represented using adjacency lists.
- (B) Merging the adjacency lists of the two vertices being contracted.
- (C) Using hashing to do fix-ups.
  (i.e., fix adjacency list of vertices connected to $x, y$.)
- (D) Include edge weight in computing cut weight.

#### 11.3.0.6 Cuts under contractions

**Observation 11.3.1.** (A) A cut in $G/xy$ is a valid cut in $G$.

(B) There $\exists$ cuts in $G$ are not in $G/xy$.

(C) The cut $S = \{x\}$ is not in $G/xy$.

(D) $\implies$ size mincut in $G/xy \geq$ mincut in $G$.

- (A) **Idea**: Repeatedly perform edge contractions (benefits: shrink graph)...
- (B) Every vertex in contracted graph is a connected component in the original graph.)
11.3.0.7 Contraction

(2) $x$  $y$

(3) $\{x, y\}$

(4)

(5)

(7)

(8)

(9)

(10)

(11)

(12)

11.3.0.8 Contraction - all together now

(a) (b) (c) (d)

(e) (f) (g) (h)

(i) (j)

11.3.0.9 But...

(A) Not min cut!
(B) Contracted wrong edge somewhere...
(C) If never contract an edge in the cut...
(D) ...get min cut in the end!
(E) We might still get min cut even if we contract edge min cut. Why???

11.3.1 The resulting algorithm

11.3.1.1 The algorithm...

\begin{algorithm}
\textbf{Algorithm} \hspace{1em} \textbf{MinCut}(G) \\
\hspace{1em} G_0 \leftarrow G \\
\hspace{1em} i = 0 \\
\hspace{2em} \textbf{while} \hspace{1em} G_i \text{ has more than two vertices} \textbf{do} \\
\hspace{3em} e_i \leftarrow \text{random edge from } E(G_i) \\
\hspace{3em} G_{i+1} \leftarrow G_i / e_i \\
\hspace{3em} i \leftarrow i + 1 \\
\hspace{2em} \textbf{Let} \hspace{1em} (S, V \setminus S) \text{ be the cut in the original graph} \\
\hspace{2em} \text{corresponding to the single edge in } G_i \\
\hspace{2em} \textbf{return} \hspace{1em} (S, V \setminus S).
\end{algorithm}

11.3.1.2 How to pick a random edge?

Lemma 11.3.2. \(X = \{x_1, \ldots, x_n\}: \text{ elements, } \omega(x_i): \text{ integer positive weight.}\)

Pick randomly, in \(O(n)\) time, an element \(x_i \in X\), with prob picking \(x_i\) being \(\omega(x_i) / W\), where \(W = \sum_{i=1}^{n} \omega(x_i)\).

\textbf{Proof}: Randomly choose \(r \in [0, W]\).

Precompute \(\beta_i = \sum_{k=1}^{i} \omega(x_k) = \beta_{i-1} + \omega(x_i)\).

Find first index \(i\), \(\beta_{i-1} < r \leq \beta_i\). Return \(x_i\).

(A) Edges have weight...
(B) ...compute total weight of each vertex (adjacent edges).
(C) Pick randomly a vertex by weight.
(D) Pick random edge adjacent to this vertex.

11.3.2 Analysis

11.3.2.1 The probability of success

11.3.2.2 Lemma...

Lemma 11.3.3. \(G: \text{ mincut of size } k \text{ and } n \text{ vertices, then } |E(G)| \geq \frac{kn}{2}\).

\textbf{Proof}: Each vertex degree is at least \(k\), otherwise the vertex itself would form a minimum cut of size smaller than \(k\). As such, there are at least \(\sum_{v \in V} \text{degree}(v)/2 \geq nk/2\) edges in the graph.

11.3.2.3 Lemma...

Lemma 11.3.4. If we pick in random an edge \(e\) from a graph \(G\), then with probability at most \(\frac{2}{n}\) it belong to the minimum cut.

\textbf{Proof}: There are at least \(nk/2\) edges in the graph and exactly \(k\) edges in the minimum cut. Thus, the probability of picking an edge from the minimum cut is smaller then \(k/(nk/2) = 2/n\).
11.3.2.4 Lemma

Lemma 11.3.5. \textbf{MinCut} outputs the mincut with prob. $\geq \frac{2}{n(n-1)}$.

Proof

(A) $\mathcal{E}_i$: event that $e_i$ is not in the minimum cut of $G_i$.

(B) \textbf{MinCut} outputs mincut if all the events $\mathcal{E}_0, \ldots, \mathcal{E}_{n-3}$ happen.

(C) $\Pr[\mathcal{E}_i \mid \mathcal{E}_0 \cap \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}] \geq 1 - \frac{2}{|V(G_i)|} = 1 - \frac{2}{n-i}$.

$\implies \Delta = \Pr[\mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{n-3}] = \Pr[\mathcal{E}_0] \cdot \Pr[\mathcal{E}_1 \mid \mathcal{E}_0] \cdot \Pr[\mathcal{E}_2 \mid \mathcal{E}_0 \cap \mathcal{E}_1] \cdot \ldots \cdot \Pr[\mathcal{E}_{n-3} \mid \mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{n-4}]$

11.3.2.5 Proof continued...

As such, we have

$$\Delta \geq \prod_{i=0}^{n-3} \left(1 - \frac{2}{n-i}\right) = \prod_{i=0}^{n-3} \frac{n-i-2}{n-i}$$

$$= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \ldots \cdot \frac{2}{4} \cdot \frac{1}{3}$$

$$= \frac{2}{n \cdot (n-1)}.$$

11.3.2.6 Running time analysis.

11.3.2.7 Running time analysis...

Observation 11.3.6. \textbf{MinCut} runs in $O(n^2)$ time.

Observation 11.3.7. The algorithm always outputs a cut, and the cut is not smaller than the minimum cut.

Definition 11.3.8. \textbf{Amplification}: running an experiment again and again till the things we want to happen, with good probability, do happen.

11.3.2.8 Getting a good probability

\textbf{MinCutRep}: algorithm runs \textbf{MinCut} $n(n-1)$ times and return the minimum cut computed.

Lemma 11.3.9. probability \textbf{MinCutRep} fails to return the minimum cut is $< 0.14$.

Proof: \textbf{MinCut} fails to output the mincut in each execution is at most $1 - \frac{2}{n(n-1)}$.

\textbf{MinCutRep} fails, only if all $n(n-1)$ executions of \textbf{MinCut} fail.

$\left(1 - \frac{2}{n(n-1)}\right)^{n(n-1)} \leq \exp\left(-\frac{2}{n(n-1)} \cdot n(n-1)\right) = \exp(-2) < 0.14$, since $1 - x \leq e^{-x}$ for $0 \leq x \leq 1$.

11.3.2.9 Result

Theorem 11.3.10. One can compute mincut in $O(n^4)$ time with constant probability to get a correct result. In $O(n^4 \log n)$ time the minimum cut is returned with high probability.
11.4 A faster algorithm

11.4.0.10 Faster algorithm

Why MinCutRep needs so many executions?

Probability of failure in first $\nu$ iterations is

$$\Pr \left[ E_0 \cap \ldots \cap E_{\nu-1} \right] \geq \prod_{i=0}^{\nu-1} \left( 1 - \frac{2}{n-i} \right) = \prod_{i=0}^{\nu-1} \frac{n-i-2}{n-i}$$

$$= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdots$$

$$= \frac{(n-\nu)(n-\nu-1)}{n \cdot (n-1)}.$$ 

$\implies \nu = n/2$: Prob of success $\approx 1/4$.

$\implies \nu = n - \sqrt{n}$: Prob of success $\approx 1/n$.

11.4.0.11 Faster algorithm...

Insight

(A) As the graph get smaller probability for bad choice increases.

(B) Currently do the amplification from the outside of the algorithm.

(C) Put amplification directly into the algorithm.

11.4.1 Contract...

11.4.1.1 Contract($G, t$) shrinks $G$ till it has only $t$ vertices. FastCut computes the minimum cut using Contract.

```plaintext
FastCut(G = (V, E))
G -- multi-graph
begin
    n ← |V(G)|
    if $n \leq 6$ then
        Compute minimum cut
        of G and return cut.
    t ← $1 + n/\sqrt{2}$
    $H_1 \leftarrow \text{Contract}(G, t)$
    $H_2 \leftarrow \text{Contract}(G, t)$
    /* Contract is randomized!!! */
    $X_1 \leftarrow \text{FastCut}(H_1)$,
    $X_2 \leftarrow \text{FastCut}(H_2)$
    return mincut of $X_1$ and $X_2$.
end
```

11.4.1.2 Lemma...

Lemma 11.4.1. The running time of FastCut($G$) is $O(n^2 \log n)$, where $n = |V(G)|$.

Proof: Well, we perform two calls to Contract($G, t$) which takes $O(n^2)$ time. And then we perform two recursive calls on the resulting graphs. We have:

$$T(n) = O(n^2) + 2T\left(\frac{n}{\sqrt{2}}\right)$$

The solution to this recurrence is $O(n^2 \log n)$ as one can easily (and should) verify. ■
11.4.1.3 Success at each step

Lemma 11.4.2. Probability that mincut in contracted graph is original mincut is at least 1/2.

Proof: Plug in $\nu = n - t = n - \left\lceil 1 + n/\sqrt{2} \right\rceil$ into success probability:

$$\Pr \left[ \mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{n-t} \right] \geq \frac{t(t-1)}{n \cdot (n-1)} = \frac{\left\lceil 1 + n/\sqrt{2} \right\rceil \left( \left\lceil 1 + n/\sqrt{2} \right\rceil - 1 \right)}{n(n-1)} \geq \frac{1}{2}.$$

11.4.1.4 Probability of success...

Lemma 11.4.3. **FastCut** finds the minimum cut with probability larger than $\Omega(1/\log n)$.

See class notes for a formal proof. We provide a more elegant direct argument shortly.

11.4.1.5 Amplification

Lemma 11.4.4. Running **FastCut** repeatedly $c \cdot \log^2 n$ times, guarantee that the algorithm outputs mincut with probability $\geq 1 - 1/n^2$.

$c$ is a constant large enough.

Proof:
(A) **FastCut** succeeds with prob $\geq c'/\log n$, $c'$ is a constant.
(B) ...fails with prob. $\leq 1 - c'/\log n$.
(C) ...fails in $m$ reps with prob. $\leq (1 - c'/\log n)^m$. But then

$$(1 - c'/\log n)^m \leq \left( e^{-c'/\log n} \right)^m \leq e^{-mc'/\log n} \leq \frac{1}{n^2},$$

for $m = (2 \log n) / c'$.

11.4.1.6 Theorem

Theorem 11.4.5. One can compute the minimum cut in a graph $G$ with $n$ vertices in $O(n^2 \log^3 n)$ time. The algorithm succeeds with probability $\geq 1 - 1/n^2$.

Proof: We do amplification on **FastCut** by running it $O(\log^2 n)$ times. The running time bound follows from lemma...

11.5 On coloring trees and min-cut

11.5.0.7 Trees and coloring edges...

(A) $T_h$ be a complete binary tree of height $h$.
(B) Randomly color its edges by black and white.
(C) $\mathcal{E}_h$: there exists a black path from root $T_h$ to one of its leafs.
(D) $\rho_h = \Pr[\mathcal{E}_h]$.
(E) $\rho_0 = 1$ and $\rho_1 = 3/4$ (see below).
11.5.0.8 Bounding $\rho_h$

(A) $u$ root of $T_h$: children $u_l$ and $u_r$.
(B) $\rho_{h-1}$: Probability for black path $u_l \leadsto$ children
(C) Prob of black path from $u$ through $u_1$ is:
   \[ \Pr[ uu_l \text{ is black}] = \rho_{h-1} \]
(D) Prob. no black path through $u_l$ is $1 - \rho_{h-1}/2$.
(E) Prob no black path is: $(1 - \rho_{h-1}/2)^2$
(F) We have
   \[
   \rho_h = 1 - \left(1 - \frac{\rho_{h-1}}{2}\right)^2 = \frac{\rho_{h-1}}{2} \left(2 - \frac{\rho_{h-1}}{2}\right) = \rho_{h-1} - \frac{\rho_{h-1}^2}{4}.
   \]

11.5.0.9 Lemma...

Lemma 11.5.1. We have that $\rho_h \geq 1/(h + 1)$.

Proof:
(A) By induction. For $h = 1$: $\rho_1 = 3/4 \geq 1/(1 + 1)$.
(B) $\rho_h = \rho_{h-1} - \frac{\rho_{h-1}^2}{4} = f(\rho_{h-1})$, for $f(x) = x - x^2/4$.
(C) $f'(x) = 1 - x/2$. \(\Rightarrow\) $f'(x) > 0$ for $x \in [0, 1]$.
(D) $f(x)$ is increasing in the range $[0, 1]$
(E) By induction: $\rho_h = f(\rho_{h-1}) \geq f \left( \frac{1}{(h-1) + 1} \right) = \frac{1}{h} - \frac{1}{4h^2}$.
(F) $\frac{1}{h} - \frac{1}{4h^2} \geq \frac{1}{h+1}$ \(\Leftrightarrow\) $4h(h+1) - (h+1) \geq 4h^2$ \(\Leftrightarrow\) $4h^2 + 4h - h - 1 \geq 4h^2$ \(\Leftrightarrow\) $3h \geq 1$.

11.5.0.10 Back to FastCut...

(A) Recursion tree for FastCut corresponds to such a coloring.
(B) Every call performs two recursive calls.
(C) Contraction in recursion succeeds with prob $1/2$.
   Draw recursion edge in black if successful.
(D) algorithm succeeds \(\iff\) there black path from root of recursion tree to leaf.
(E) Since depth of tree $H \leq 2 + \log_{\sqrt{2}} n$.
(F) by above... probability of success is $\geq 1/(h + 1) \geq 1/(3 + \log_{\sqrt{2}} n)$.

11.5.0.11 Galton-Watson processes

(A) Start with a single node.
(B) Each node has two children.
(C) Each child survives with probability half (independently).
(D) If a child survives then it is going to have two children, and so on.
(E) A single node give a rise to a random tree.
(F) Q: Probability that the original node has descendants $h$ generations in the future.
(G) Prove this probability is at least $1/(h + 1)$.
11.5.0.12 Galton-Watson process

(A) Victorians worried: aristocratic surnames were disappearing.
(B) Family names passed on only through the male children.
(C) Family with no male children had its family name disappear.
(D) \# male children of a person is an independent random variable \( X \in \{0, 1, 2, \ldots \} \).
(E) Starting with a single person, its family (as far as male children are concerned) is a random tree
   with the degree of a node being distributed according to \( X \).
(F) .. A family disappears if \( E[X] \leq 1 \), and it has a constant probability of surviving if \( E[X] > 1 \).

11.5.0.13 Galton-Watson process

(A) ... Infant mortality is dramatically down. No longer a problem.
(B) Countries with family names that were introduced long time ago...
(C) ...have very few surnames.
   Koreans have 250 surnames, and three surnames form 45% of the population).
(D) Countries introduced surnames recently have more surnames.
   Dutch have surnames only for the last 200 years, and there are 68,000 different family names).