Approximation Algorithms

Lecture 7

September 17, 2013
Don’t give up on **NP-Hard** problems:

(A) Faster exponential time algorithms: $n^{O(n)}$, $3^n$, $2^n$, etc.

(B) Fixed parameter tractable.

(C) Find an approximate solution.
Part I

Greedy algorithms and approximation algorithms
Greedy algorithms

(A) *greedy algorithms*: do locally the right thing...

(B) ...and they suck.

**VertexCoverMin**

**Instance:** A graph $G$.

**Question:** Return the smallest subset $S \subseteq V(G)$, s.t. $S$ touches all the edges of $G$.

(C) **GreedyVertexCover:**

pick vertex with highest degree, remove, repeat.

(D) Returns 4, but opt is 3!
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Good enough...

Definition

In a *minimization* optimization problem, one looks for a valid solution that minimizes a certain target function.

1. **VertexCoverMin**: $\text{Opt}(G) = \min_{S \subseteq V(G), S \text{ cover of } G} |S|$.
2. **VertexCover**$(G)$: set realizing sol.
3. **Opt**$(G)$: value of the target function for the optimal solution.
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Definition

**Alg** is *\(\alpha\)-approximation algorithm* for problem **Min**, achieving an approximation \(\alpha \geq 1\), if for all inputs \(G\), we have:

\[
\frac{\text{Alg}(G)}{\text{Opt}(G)} \leq \alpha.
\]
Our **GreedyVertexCover** Example

(A) **GreedyVertexCover**: pick vertex with highest degree, remove, repeat.

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(C) Can not be better than a 4/3-approximation algorithm.

(D) Actually it is much worse!
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How bad is \textbf{GreedyVertexCover}?

Build a bipartite graph.

Let the top partite set be of size \( n \).
How bad is \textbf{GreedyVertexCover}?

Build a bipartite graph.

In the bottom set add \(\lfloor n/2 \rfloor\) vertices of degree 2, such that each edge goes to a different vertex above.
How bad is GreedyVertexCover?

Build a bipartite graph.

Repeatedly add \( \lfloor n/i \rfloor \) bottom vertices of degree \( i \), for \( i = 2, \ldots, n \).
How bad is GreedyVertexCover?

Build a bipartite graph.

Repeatedly add $\left\lfloor \frac{n}{i} \right\rfloor$ bottom vertices of degree $i$, for $i = 2, \ldots, n$. 
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Repeatedly add \( \lfloor n/i \rfloor \) bottom vertices of degree \( i \), for \( i = 2, \ldots, n \).
How bad is **GreedyVertexCover**?

Build a bipartite graph.

Bottom row has $\sum_{i=2}^{n} \lfloor n/i \rfloor = \Theta(n \log n)$ vertices.
How bad is **GreedyVertexCover**?
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1. Bottom row taken by Greedy.
How bad is `GreedyVertexCover`?

1. Bottom row taken by Greedy.
2. Top row was a smaller solution.
How bad is \textbf{GreedyVertexCover}? 

1. Bottom row taken by Greedy.
2. Top row was a smaller solution.

\textbf{Lemma}

The algorithm \textbf{GreedyVertexCover} is $\Omega(\log n)$ approximation to the optimal solution to \textbf{VertexCoverMin}.

See notes for details!
Greedy Vertex Cover

**Theorem**

The greedy algorithm for *VertexCover* achieves $\Theta(\log n)$ approximation, where $n$ (resp. $m$) is the number of vertices (resp., edges) in the graph. Its running time is $O(mn^2)$.

**Proof**

Lower bound follows from lemma.
Upper bound follows from analysis of greedy algorithm for *Set Cover*, which will be done shortly.
As for the running time, each iteration of the algorithm takes $O(mn)$ time, and there are at most $n$ iterations.
ApproxVertexCover(G):

\[
S \leftarrow \emptyset \\
\text{while } E(G) \neq \emptyset \text{ do} \\
\quad uv \leftarrow \text{any edge of } G \\
\quad S \leftarrow S \cup \{u, v\} \\
\quad \text{Remove } u, v \text{ from } V(G) \\
\quad \text{Remove all edges involving } u \text{ or } v \text{ from } E(G)
\]

return \( S \)

Theorem

ApproxVertexCover is a 2-approximation algorithm for VertexCoverMin that runs in \( O(n^2) \) time.

Proof...
Part II

Fixed parameter tractability, approximation, and fast exponential time algorithms (to say nothing of the dog)
What if the vertex cover is small?

1. $G = (V, E)$ with $n$ vertices
2. $K \leftarrow$ Approximate $\text{VertexCoverMin}$ up to a factor of two.
3. Any vertex cover of $G$ is of size $\geq K/2$.
4. Naively compute optimal in $O(n^{K+2})$ time.
**Induced subgraph**

**Definition**

\[ N_G(v) : \textit{Neighborhood} \text{ of } v \]

– set of vertices of \( G \) adjacent to \( v \).
**Induced subgraph**

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\( N_G(v) \): *Neighborhood* of \( v \) – set of vertices of \( G \) adjacent to \( v \).

Let \( G = (V, E) \) be a graph. For a subset \( S \subseteq V \), let \( G_S \) be the *induced subgraph* over \( S \).
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Definition

$N_G(v)$: *Neighborhood* of $v$ – set of vertices of $G$ adjacent to $v$.

Definition

Let $G = (V, E)$ be a graph. For a subset $S \subseteq V$, let $G_S$ be the *induced subgraph* over $S$. 
Computes minimum vertex cover for the induced graph $G_X$:

```plaintext
fpVCI(X, β)
// β: size of VC computed so far.
if X = ∅ or $G_X$ has no edges then return β

\[ e \leftarrow \text{any edge } uv \text{ of } G_X. \]
\[ β_1 = fpVCI(X \setminus \{u, v\}, β + 2) \]
\[ β_2 = fpVCI(X \setminus (\{u\} \cup N_{G_X}(v)), β + |N_{G_X}(v)|) \]
\[ β_3 = fpVCI(X \setminus (\{v\} \cup N_{G_X}(u)), β + |N_{G_X}(u)|) \]
return min(β₁, β₂, β₃).
```

```
algFPVertexCover(G = (V, E))
return fpVCI(V, 0)
```
Depth of recursion

**Lemma**

The algorithm $\text{algFPVertexCover}$ returns the optimal solution to the given instance of $\text{VertexCoverMin}$.

**Proof...**
Lemma

The depth of the recursion of $\text{algFPVertexCover}(G)$ is at most $\alpha$, where $\alpha$ is the minimum size vertex cover in $G$.

Proof.

1. When the algorithm takes both $u$ and $v$ - one of them in opt. Can happen at most $\alpha$ times.
2. Algorithm picks $N_{G_x}(v)$ (i.e., $\beta_2$). Conceptually add $v$ to the vertex cover being computed.
3. Do the same thing for the case of $\beta_3$.
4. Every such call add one element of the opt to conceptual set cover. Depth of recursion is $\leq \alpha$. 
Vertex Cover

Exact fixed parameter tractable algorithm

Theorem

Let $G$ be a graph with $n$ vertices. Min vertex cover of size $\alpha$. Then, $\text{algFPVertexCover}$ returns opt. vertex cover. Running time is $O(3^\alpha n^2)$.

Proof:

1. By lemma, recursion tree has depth $\alpha$.
2. Rec-tree contains $\leq 2 \cdot 3^\alpha$ nodes.
3. Each node requires $O(n^2)$ work.

Algorithms with running time $O(n^c f(\alpha))$, where $\alpha$ is some parameter that depends on the problem are fixed parameter tractable.
Vertex Cover

Exact fixed parameter tractable algorithm

**Theorem**

**G:** graph with \( n \) vertices. Min vertex cover of size \( \alpha \). Then, \texttt{algFPVertexCover} returns opt. vertex cover. Running time is \( O(3^\alpha n^2) \).

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Part III

Traveling Salesperson Problem
TSP-Min

**Instance:** $G = (V, E)$ a complete graph, and $\omega(e)$ a cost function on edges of $G$.

**Question:** The cheapest tour that visits all the vertices of $G$ exactly once.

Solved exactly naively in $\approx n!$ time.
Using DP, solvable in $O(n^22^n)$ time.
**TSP-Min**

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Theorem

*TSP-Min* can not be approximated within *any* factor unless \( \text{NP} = \text{P} \).

Proof.

1. Reduction from **Hamiltonian Cycle** into **TSP**.
2. \( G = (V, E) \): instance of Hamiltonian cycle.
3. \( H \): Complete graph over \( V \).
   \[
   \forall u, v \in V \quad w_H(uv) = \begin{cases} 
   1 & uv \in E \\
   2 & \text{otherwise}
   \end{cases}
   \]
4. \( \exists \) tour of price \( n \) in \( H \) \( \iff \) \( \exists \) Hamiltonian cycle in \( G \).
5. No Hamiltonian cycle \( \implies \) TSP price at least \( n + 1 \).
6. But... replace \( 2 \) by \( cn \), for \( c \) an arbitrary number
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TSP Hardness

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Proof.

1. Price of all tours are either:
   (i) $n$ (only if $\exists$ Hamiltonian cycle in $G$),
   (ii) larger than $cn + 1$ (actually, $\geq cn + (n - 1)$).

2. Suppose you had a poly time $c$-approximation to TSP-Min.

3. Run it on $H$:
   (i) If returned value $\geq cn + 1 \implies$ no Ham Cycle since $(cn + 1)/c > n$
   (ii) If returned value $\leq cn \implies$ Ham Cycle since $OPT \leq cn < cn + 1$

4. $c$-approximation algorithm to TSP $\implies$ poly-time algorithm for NP-Complete problem. Possible only if $P = NP$. 

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CS573 21
Fall 2013 21 / 35
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TSP Hardness - proof continued

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TSP with the triangle inequality

Because it is not that bad after all.

**TSP\(\bigtriangleup \neq\)-Min**

**Instance:** \(G = (V, E)\) is a complete graph. There is also a cost function \(\omega(\cdot)\) defined over the edges of \(G\), that complies with the triangle inequality.

**Question:** The cheapest tour that visits all the vertices of \(G\) exactly once.

*triangle inequality:* \(\omega(\cdot)\) if

\[
\forall u, v, w \in V(G), \quad \omega(u, v) \leq \omega(u, w) + \omega(w, v).
\]

**Shortcutting**

\(\sigma\): a path from \(s\) to \(t\) in \(G\) \(\implies \omega(st) \leq \omega(\sigma)\).
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Definition

Cycle in $G$ is **Eulerian** if it visits every edge of $G$ exactly once.

Assume you already seen the following:

Lemma

A graph $G$ has a cycle that visits every edge of $G$ exactly once (i.e., an Eulerian cycle) if and only if $G$ is connected, and all the vertices have even degree. Such a cycle can be computed in $O(n + m)$ time, where $n$ and $m$ are the number of vertices and edges of $G$, respectively.
1. $C_{opt}$ optimal TSP tour in $G$.

2. Observation:
   \[ \omega(C_{opt}) \geq \text{weight(cheapest spanning graph of } G) \].

3. MST: cheapest spanning graph of $G$.
   \[ \omega(C_{opt}) \geq \omega(\text{MST}(G)) \]

4. $O(n \log n + m) = O(n^2)$: time to compute MST.
   
   $n = |V(G)|$, $m = \binom{n}{2}$. 
**TSP with the triangle inequality**

Continued...

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CS573  
Fall 2013  
24 / 35
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TSP with the triangle inequality

2-approximation

1. \( T \leftarrow \text{MST}(G) \)
2. \( H \leftarrow \text{duplicate very edge of } T \).
3. \( H \) has an Eulerian tour.
4. \( C \): Eulerian cycle in \( H \).
5. \( \omega(C) = \omega(H) = 2\omega(T) = 2\omega(\text{MST}(G)) \leq 2\omega(C_{\text{opt}}) \).
6. \( \pi \): Shortcut \( C \) so visit every vertex once.
7. \( \omega(\pi) \leq \omega(C) \)
**TSP with the triangle inequality**

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2-approximation algorithm in figures
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(a) (b) (c) (d)
**TSP with the triangle inequality**

2-approximation algorithm in figures

(a) (b) (c) (d)

Euler Tour: **VUVWVSV**
First occurrences: **VUVWVSV**
Shortcut String: **VUWSV**
Theorem

\( G \): Instance of TSP\(_\triangle \neq \text{Min} \).
\( C_{\text{opt}} \): min cost TSP tour of \( G \).

\[ \implies \text{Compute a tour of } G \text{ of length } \leq 2\omega(C_{\text{opt}}). \]

Running time of the algorithm is \( O(n^2) \).

\( G \): \( n \) vertices, cost function \( \omega(\cdot) \) on the edges that comply with the triangle inequality.
**TSP with the triangle inequality**

2-approximation - result

---

**Theorem**

\( G \): Instance of \( TSP_{\triangle \neq Min} \).

\( C_{opt} \): min cost TSP tour of \( G \).

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TSP with the triangle inequality

$3/2$-approximation

**Definition**

$G = (V, E)$, a subset $M \subseteq E$ is a **matching** if no pair of edges of $M$ share endpoints.

A **perfect matching** is a matching that covers all the vertices of $G$.

$w$: weight function on the edges. **Min-weight perfect matching**, is the minimum weight matching among all perfect matching, where

$$\omega(M) = \sum_{e \in M} \omega(e).$$
TSP with the triangle inequality

3/2-approximation

The following is known:

**Theorem**

*Given a graph $G$ and weights on the edges, one can compute the min-weight perfect matching of $G$ in polynomial time.*
Min weight perfect matching vs. TSP

Lemma

\[ G = (V, E): \text{complete graph.} \]
\[ S \subseteq V: \text{even size.} \]
\[ \omega(\cdot): \text{a weight function over } E. \]
\[ \implies \text{min-weight perfect matching in } G_S \text{ is } \leq \frac{\omega(\text{TSP}(G))}{2}. \]
Lemma

\( G = (V, E) \): complete graph.

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A more perfect tree?

1. How to make the tree Eulerian?

   ![Diagram of a tree with vertices and edges labeled 1 to 7]

2. Pesky odd degree vertices must die!

3. Number of odd degree vertices in a graph is even!

4. Compute min-weight matching on odd vertices, and add to MST.

5. \( H = \text{MST} + \text{min-weight matching} \) is Eulerian.

6. Weight of resulting cycle in \( H \leq (3/2)\omega(\text{TSP}) \).
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Even number of odd degree vertices

Lemma

The number of odd degree vertices in any graph \( G' \) is even.

Proof:

\[
\mu = \sum_{v \in V(G')} d(v) = 2 |E(G')| \quad \text{and thus even.}
\]

\[
U = \sum_{v \in V(G'), d(v) \text{ is even}} d(v) \quad \text{even too.}
\]

Thus,

\[
\alpha = \sum_{v \in V, d(v) \text{ is odd}} d(v) = \mu - U = \text{even number,}
\]

since \( \mu \) and \( U \) are both even.

Number of elements in sum of all odd numbers must be even, since the total sum is even.
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Let $\mu = \sum_{v \in V(G')} d(v) = 2|E(G')|$ and thus even. Let $U = \sum_{v \in V(G') \mid d(v) \text{ is even}} d(v)$ be even too.

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$$\alpha = \sum_{v \in V \mid d(v) \text{ is odd}} d(v) = \mu - U = \text{even number},$$

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3/2-approximation algorithm for TSP

Animated!
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\(3/2\)-approximation algorithm for TSP
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Animated!
The result

**Theorem**

*Given an instance of TSP with the triangle inequality, one can compute in polynomial time, a \((3/2)\)-approximation to the optimal TSP.*
The $3/2$-approximation for TSP with the triangle inequality is due to Christofides [1976].
