CS 573: Algorithms, Fall 2013

Approximation Algorithms

Lecture 7

September 17, 2013

Today's Lecture

Don't give up on **NP-Hard** problems:

(A) Faster exponential time algorithms: $n^{O(n)}$, 3^n, 2^n, etc.

(B) Fixed parameter tractable.

(C) Find an approximate solution.

Part I

Greedy algorithms and approximation algorithms

Greedy algorithms

(A) ***greedy algorithms***: do locally the right thing...

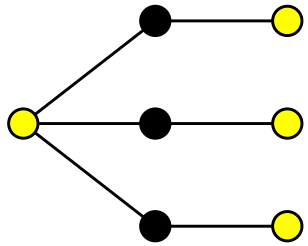
(B) ...and they suck.

VertexCoverMin

Instance: A graph $\mathbf{G}$.
Question: Return the smallest subset $S \subseteq V(\mathbf{G})$, s.t. S touches all the edges of $\mathbf{G}$.

(C) **GreedyVertexCover**: pick vertex with highest degree, remove, repeat.

(D) Returns 4, but opt is 3!



Good enough...

Definition

In a ***minimization*** optimization problem, one looks for a valid solution that minimizes a certain target function.

1. VertexCoverMin: $\mathrm{Opt}(\mathbf{G}) = \min_{S \subseteq V(\mathbf{G}), S \text{ cover of } G} |S|$.
2. $\mathrm{VertexCover}(\mathbf{G})$: set realizing sol.
3. $\mathrm{Opt}(\mathbf{G})$: value of the target function for the optimal solution.

Definition

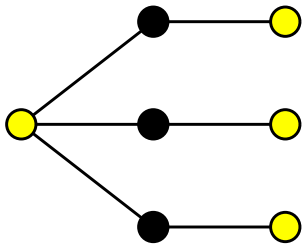
Alg is ***α-approximation algorithm*** for problem **Min**, achieving an approximation $\alpha \geq 1$, if for all inputs $\mathbf{G}$, we have:

$$\frac{\mathrm{Alg}(\mathbf{G})}{\mathrm{Opt}(\mathbf{G})} \leq \alpha.$$

Our GreedyVertexCover Example

(A) **GreedyVertexCover**: pick vertex with highest degree, remove, repeat.

(B) Returns 4, but opt is 3!



(C) Can **not** be better than a $4/3$-approximation algorithm.

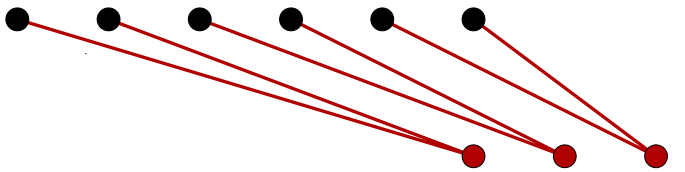
(D) Actually it is much worse!

How bad is GreedyVertexCover?

Build a bipartite graph.

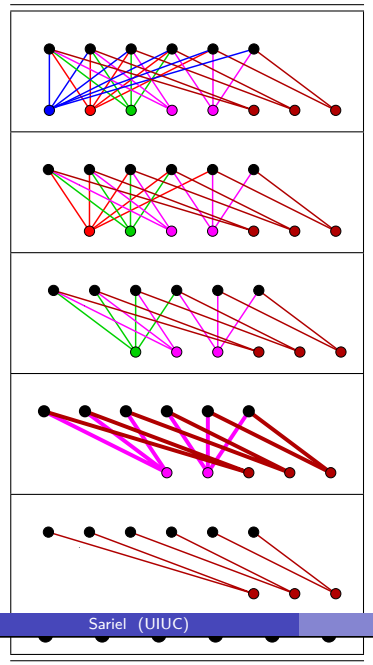
Let the top partite set be of size n.

In the bottom set add $\lfloor n/2 \rfloor$ vertices of degree 2, such that each edge goes to a different vertex above.



Repeatedly add $\lfloor n/i \rfloor$ bottom vertices of degree i, for $i = 2$ n

How bad is GreedyVertexCover?



1. Bottom row taken by Greedy.
2. Top row was a smaller solution.

Lemma

The algorithm **GreedyVertexCover** *is* $\Omega(\log n)$ *approximation to the optimal solution to VertexCoverMin.*

See notes for details!

Greedy Vertex Cover

Theorem

The greedy algorithm for ***VertexCover*** *achieves* $\Theta(\log n)$ *approximation, where* n *(resp.* m*) is the number of vertices (resp., edges) in the graph. Its running time is* $O(mn^2)$.

Proof

Lower bound follows from lemma.
Upper bound follows from analysis of greedy algorithm for **Set Cover**, which will be done shortly.
As for the running time, each iteration of the algorithm takes $O(mn)$ time, and there are at most n iterations.

Two for the price of one

```
ApproxVertexCover(G):
    S ← ∅
    while E(G) ≠ ∅ do
        uv ← any edge of G
        S ← S ∪ {u, v}
        Remove u, v from V(G)
        Remove all edges involving u or v from E(G)

    return S
```

Theorem

ApproxVertexCover *is a* 2*-approximation algorithm for VertexCoverMin that runs in* $O(n^2)$ *time.*

Proof...

Part II

Fixed parameter tractability, approximation, and fast exponential time algorithms (to say nothing of the dog)

What if the vertex cover is small?

1. $G = (V, E)$ with n vertices
2. $K \leftarrow$ Approximate VertexCoverMin up to a factor of two.
3. Any vertex cover of G is of size $\geq K/2$.
4. Naively compute optimal in $O\left(n^{K+2}\right)$ time.

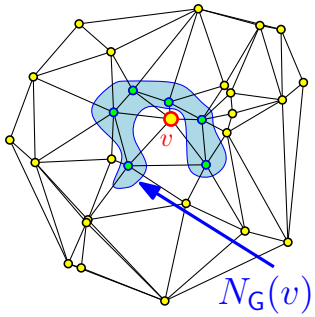
Induced subgraph

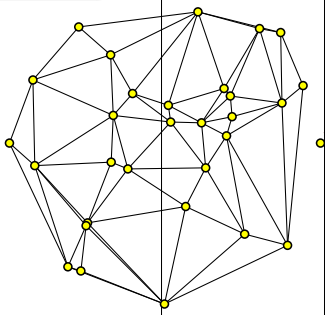
Definition

$N_G(v)$: ***Neighborhood*** of v – set of vertices of G adjacent to v.

Definition

Let $G = (V, E)$ be a graph. For a subset $S \subseteq V$, let G_S be the ***induced subgraph*** over S.





Exact fixed parameter tractable algorithm

Fixed parameter tractable algorithm for **VertexCoverMin**.

Computes minimum vertex cover for the induced graph G_X:

fpVCI(X, β)

```
// β: size of VC computed so far.
```

if $X = \emptyset$ or G_X has no edges **then return** β

$e \leftarrow$ any edge uv of G_X.

$\beta_1 = \text{fpVCI}\left(X \setminus \{u, v\}, \beta + 2\right)$

$\beta_2 = \text{fpVCI}\left(X \setminus \left(\{u\} \cup N_{G_X}(v)\right), \beta + |N_{G_X}(v)|\right)$

$\beta_3 = \text{fpVCI}\left(X \setminus \left(\{v\} \cup N_{G_X}(u)\right), \beta + |N_{G_X}(u)|\right)$

return $\min(\beta_1, \beta_2, \beta_3)$.

algFPVertexCover$(G = (V, E))$

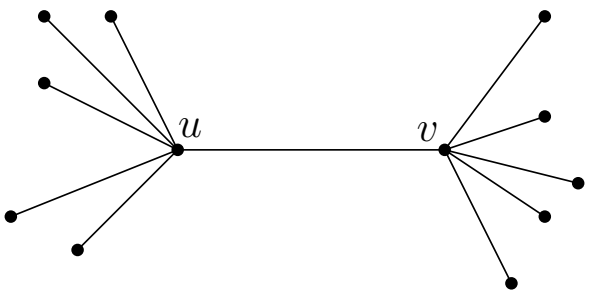
return **fpVCI**$(V, 0)$

Depth of recursion

Lemma

The algorithm **algFPVertexCover** *returns the optimal solution to the given instance of VertexCoverMin.*

Proof...



Depth of recursion II

Lemma

The depth of the recursion of **algFPVertexCover**(G) *is at most* α, *where* α *is the minimum size vertex cover in* G.

Proof.

1. When the algorithm takes both u and v - one of them in opt. Can happen at most α times.
2. Algorithm picks $N_{G_X}(v)$ (i.e., β_2). Conceptually add v to the vertex cover being computed.
3. Do the same thing for the case of β_3.
4. Every such call add one element of the opt to conceptual set cover. Depth of recursion is $\leq \alpha$.

□

Vertex Cover

Exact fixed parameter tractable algorithm

Theorem

$\mathbf{G}$: *graph with* n *vertices. Min vertex cover of size* α. *Then,* **algFPVertexCover** *returns opt. vertex cover.*
Running time is $O\left(3^{\alpha} n^2\right)$.

Proof:

1. By lemma, recursion tree has depth α.
2. Rec-tree contains $\leq 2 \cdot 3^{\alpha}$ nodes.
3. Each node requires $O\left(n^2\right)$ work. ■

Algorithms with running time $O(n^c f(\alpha))$, where α is some parameter that depends on the problem are ***fixed parameter tractable***.

Part III

Traveling Salesperson Problem

TSP

TSP-Min

Instance: $\mathbf{G} = (V, E)$ a complete graph, and $\omega(e)$ a cost function on edges of $\mathbf{G}$.
Question: The cheapest tour that visits all the vertices of $\mathbf{G}$ exactly once.

Solved exactly naively in $\approx n!$ time.
Using DP, solvable in $O(n^2 2^n)$ time.

TSP Hardness

Theorem

TSP-Min can not be approximated within ***any*** *factor unless* $\mathbf{NP} = \mathbf{P}$.

Proof.

1. Reduction from **Hamiltonian Cycle** into **TSP**.
2. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$: instance of Hamiltonian cycle.
3. $\mathbf{H}$: Complete graph over $\mathbf{V}$.
$$\forall u, v \in \mathbf{V} \quad w_{\mathbf{H}}(uv) = \begin{cases} 1 & uv \in \mathbf{E} \\ 2 & \text{otherwise.} \end{cases}$$
4. $\exists$ tour of price n in $\mathbf{H}$ $\iff$ $\exists$ Hamiltonian cycle in $\mathbf{G}$.
5. No Hamiltonian cycle $\implies$ TSP price at least $n + 1$.
6. But... replace 2 by cn, for c an arbitrary number

TSP Hardness - proof continued

Proof.

1. Price of all tours are either:
   (i) n (only if $\exists$ Hamiltonian cycle in $\mathbf{G}$),
   (ii) larger than $cn + 1$ (actually, $\geq cn + (n-1)$).
2. Suppose you had a poly time c-approximation to TSP-Min.
3. Run it on $\mathbf{H}$:
   (i) If returned value $\geq cn + 1 \implies$ no Ham Cycle since $(cn+1)/c > n$
   (ii) If returned value $\leq cn \implies$ Ham Cycle since $OPT \leq cn < cn + 1$
4. c-approximation algorithm to TSP $\implies$ poly-time algorithm for **NP-Complete** problem. Possible only if $\mathbf{P} = \mathbf{NP}$.

$\square$

TSP with the triangle inequality

Because it is not that bad after all.

TSP$_{\triangle\neq}$-Min

Instance: $\mathbf{G} = (V, E)$ is a complete graph. There is also a cost function $\omega(\cdot)$ defined over the edges of $\mathbf{G}$, that complies with the triangle inequality.
Question: The cheapest tour that visits all the vertices of $\mathbf{G}$ exactly once.

triangle inequality: $\omega(\cdot)$ if

$$\forall u, v, w \in \mathbf{V}(\mathbf{G}), \qquad \omega(u, v) \leq \omega(u, w) + \omega(w, v).$$

Shortcutting

σ: a path from s to t in $\mathbf{G} \implies \omega(st) \leq \omega(\sigma)$.

TSP with the triangle inequality

Continued...

Definition

Cycle in $\mathbf{G}$ is ***Eulerian*** if it visits every edge of $\mathbf{G}$ exactly once.

Assume you already seen the following:

Lemma

A graph $\mathbf{G}$ has a cycle that visits every edge of $\mathbf{G}$ exactly once (i.e., an Eulerian cycle) if and only if $\mathbf{G}$ is connected, and all the vertices have even degree. Such a cycle can be computed in $O(n + m)$ time, where n and m are the number of vertices and edges of $\mathbf{G}$, respectively.

TSP with the triangle inequality

Continued...

1. C_{opt} optimal **TSP** tour in $\mathbf{G}$.
2. **Observation**:
   $\omega(C_{\mathrm{opt}}) \geq \mathrm{weight}\big(\text{cheapest spanning graph of } \mathbf{G}\big)$.
3. MST: cheapest spanning graph of $\mathbf{G}$.
   $\omega(C_{\mathrm{opt}}) \geq \omega(\mathrm{MST}(\mathbf{G}))$
4. $O(n \log n + m) = O(n^2)$: time to compute MST.
   $n = |\mathbf{V}(\mathbf{G})|$, $m = \binom{n}{2}$.

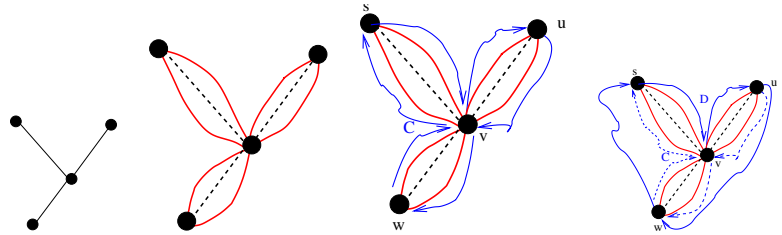
TSP with the triangle inequality

2-approximation

1. $T \leftarrow \mathrm{MST}(\mathbf{G})$
2. $\mathbf{H} \leftarrow$ duplicate very edge of T.
3. H has an Eulerian tour.
4. $\mathbf{C}$: Eulerian cycle in H.
5. $\omega(\mathbf{C}) = \omega(H) = 2\omega(T) = 2\omega(MST(\mathbf{G})) \leq 2\omega(C_{\mathrm{opt}})$.
6. π: Shortcut $\mathbf{C}$ so visit every vertex once.
7. $\omega(\pi) \leq \omega(\mathbf{C})$

TSP with the triangle inequality

2-approximation algorithm in figures



(a) (b) (c) (d)

Euler Tour: vuvwvsv
First occurrences: vuvwvsv
Shortcut String: vuwsv

TSP with the triangle inequality

2-approximation - result

Theorem

$\mathbf{G}$: *Instance of* $TSP_{\triangle\neq}$*-Min.*
C_{opt}: *min cost TSP tour of* $\mathbf{G}$.
$\Longrightarrow$ *Compute a tour of* $\mathbf{G}$ *of length* $\leq 2\omega(C_{\mathrm{opt}})$.
Running time of the algorithm is $O\left(n^2\right)$.

$\mathbf{G}$: n vertices, cost function $\omega(\cdot)$ on the edges that comply with the triangle inequality.

TSP with the triangle inequality

3/2-approximation

Definition

$\mathbf{G} = (V, E)$, a subset $M \subseteq E$ is a ***matching*** if no pair of edges of M share endpoints.
A ***perfect matching*** is a matching that covers all the vertices of $\mathbf{G}$.
w: weight function on the edges. ***Min-weight perfect matching***, is the minimum weight matching among all perfect matching, where

$$\omega(M) = \sum_{e \in M} \omega(e)\,.$$

TSP with the triangle inequality

3/2-approximation

The following is known:

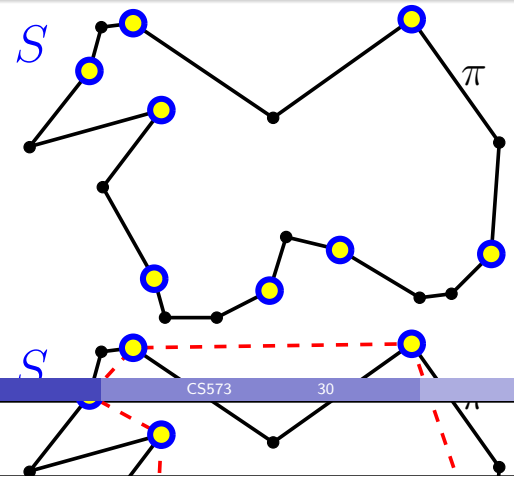
Theorem

Given a graph $\mathbf{G}$ *and weights on the edges, one can compute the min-weight perfect matching of* $\mathbf{G}$ *in polynomial time.*

Min weight perfect matching vs. TSP

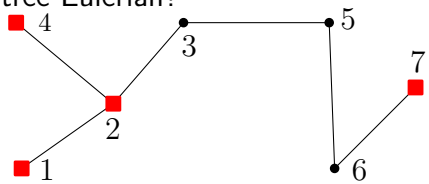
Lemma

$\mathbf{G} = (\mathbf{V}, \mathbf{E})$: *complete graph.*
$S \subseteq \mathbf{V}$: *even size.*
$\omega(\cdot)$: *a weight function over* $\mathbf{E}$.
$\implies$ *min-weight perfect matching in* $\mathbf{G}_S$ *is* $\leq \omega(\mathrm{TSP}(\mathbf{G}))/2$.



A more perfect tree?

1. How to make the tree Eulerian?



2. Pesky odd degree vertices must die!
3. Number of odd degree vertices in a graph is even!
4. Compute min-weight matching on odd vertices, and add to MST.
5. $\mathbf{H} = \mathrm{MST} + \mathbf{min-weight-matching}$ is Eulerian.
6. Weight of resulting cycle in $\mathbf{H} \leq (3/2)\omega(\mathrm{TSP})$.

Even number of odd degree vertices

Lemma

The number of odd degree vertices in any graph G' *is even.*

Proof:

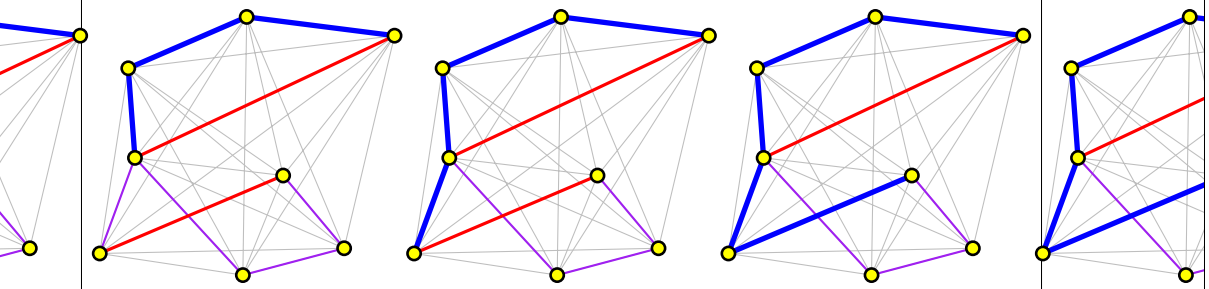
$\mu = \sum_{v \in V(G')} d(v) = 2|E(G')|$ and thus even.
$U = \sum_{v \in V(G'), d(v) \text{ is even}} d(v)$ even too.
Thus,

$$\alpha = \sum_{v \in V, d(v) \text{ is odd}} d(v) = \mu - U = \text{even number},$$

since μ and U are both even.
Number of elements in sum of all odd numbers must be even, since the total sum is even.

3/2-approximation algorithm for TSP

Animated!



3/2-approximation algorithm for TSP

The result

Theorem

Given an instance of TSP with the triangle inequality, one can compute in polynomial time, a **(3/2)***-approximation to the optimal TSP.*

Biographical Notes

The **3/2**-approximation for TSP with the triangle inequality is due to Christofides [1976].

N. Christofides. Worst-case analysis of a new heuristic for the travelling salesman problem. Technical Report Report 388, Graduate School of Industrial Administration, Carnegie Mellon University, 1976.

U. Feige and M. Goemans. Approximating the value of two power proof systems, with applications to max 2sat and max dicut. In *ISTCS '95: Proceedings of the 3rd Israel Symposium on the Theory of Computing Systems (ISTCS'95)*, page 182, Washington, DC, USA, 1995. IEEE Computer Society. ISBN 0-8186-6915-2.

J. Håstad. Some optimal inapproximability results. *J. Assoc. Comput. Mach.*, 48(4):798–859, July 2001. ISSN 0004-5411.