Part I

NP Completeness
Certifiers

**Definition**

An algorithm $C(\cdot, \cdot)$ is a **certifier** for problem $X$ if for every $s \in X$ there is some string $t$ such that $C(s, t) = \text{"yes"}$, and conversely, if for some $s$ and $t$, $C(s, t) = \text{"yes"}$ then $s \in X$. The string $t$ is called a **certificate** or **proof** for $s$.

**Definition (Efficient Certifier.)**

A certifier $C$ is an **efficient certifier** for problem $X$ if there is a polynomial $p(\cdot)$ such that for every string $s$, we have that

1. $s \in X$ if and only if
2. there is a string $t$:
   1. $|t| \leq p(|s|)$,
   2. $C(s, t) = \text{"yes"}$,
   3. and $C$ runs in polynomial time.
NP-Complete Problems

Definition

A problem $X$ is said to be **NP-Complete** if

1. $X \in \text{NP}$, and
2. (Hardness) For any $Y \in \text{NP}$, $Y \leq_P X$. 

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Proposition

Suppose $X$ is NP-Complete. Then $X$ can be solved in polynomial time if and only if $P = NP$.

Proof.

$\Rightarrow$ Suppose $X$ can be solved in polynomial time

1. Let $Y \in NP$. We know $Y \leq_P X$.
2. We showed that if $Y \leq_P X$ and $X$ can be solved in polynomial time, then $Y$ can be solved in polynomial time.
3. Thus, every problem $Y \in NP$ is such that $Y \in P$; $NP \subseteq P$.
4. Since $P \subseteq NP$, we have $P = NP$.

$\Leftarrow$ Since $P = NP$, and $X \in NP$, we have a polynomial time algorithm for $X$. 

\[\square\]
## NP-Hard Problems

### Definition

A problem $X$ is said to be **NP-Hard** if

1. **(Hardness)** For any $Y \in \text{NP}$, we have that $Y \leq_p X$.

An **NP-Hard** problem need not be in $\text{NP}$!

### Example

Example: Halting problem is **NP-Hard** (why?) but not **NP-Complete**.
Consequences of proving **NP-Completeness**

If $X$ is **NP-Complete**

1. Since we believe $P \neq NP$,
2. and solving $X$ implies $P = NP$.

$X$ is unlikely to be efficiently solvable.

At the very least, many smart people before you have failed to find an efficient algorithm for $X$.

(This is proof by mob opinion — take with a grain of salt.)
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Question

Are there any problems that are NP-Complete?

Answer

Yes! Many, many problems are NP-Complete.
Circuits

Definition

A circuit is a directed *acyclic* graph with

1. **Input** vertices (without incoming edges) labelled with 0, 1 or a distinct variable.
2. Every other vertex is labelled ∨, ∧ or ¬.
3. Single node **output** vertex with no outgoing edges.
Cook-Levin Theorem

Definition (Circuit Satisfaction (CSAT).)
Given a circuit as input, is there an assignment to the input variables that causes the output to get value $1$?

Theorem (Cook-Levin)

*CSAT* is NP-Complete.

Need to show
1. *CSAT* is in NP.
2. Every NP problem $X$ reduces to CSAT.
Claim

$\text{CSAT} \text{ is in } \text{NP}$.

1. **Certificate:** Assignment to input variables.
2. **Certifier:** Evaluate the value of each gate in a topological sort of DAG and check the output gate value.
Claim

\textit{CSAT} is in \textbf{NP}.

1. **Certificate:** Assignment to input variables.
2. **Certifier:** Evaluate the value of each gate in a topological sort of \textbf{DAG} and check the output gate value.
CSAT is NP-hard: Idea

Need to show that every NP problem $X$ reduces to CSAT.

What does it mean that $X \in \text{NP}$?

$X \in \text{NP}$ implies that there are polynomials $p()$ and $q()$ and certifier/verifier program $C$ such that for every string $s$ the following is true:

1. If $s$ is a YES instance ($s \in X$) then there is a proof $t$ of length $p(|s|)$ such that $C(s, t)$ says YES.
2. If $s$ is a NO instance ($s \notin X$) then for every string $t$ of length at $p(|s|)$, $C(s, t)$ says NO.
3. $C(s, t)$ runs in time $q(|s| + |t|)$ time (hence polynomial time).
Reducing $X$ to CSAT

$X$ is in **NP** means we have access to $p(), q(), C(\cdot, \cdot)$. What is $C(\cdot, \cdot)$? It is a program or equivalently a Turing Machine!

How are $p()$ and $q()$ given? As numbers.

Example: if 3 is given then $p(n) = n^3$.

Thus an **NP** problem is essentially a three tuple $\langle p, q, C \rangle$ where $C$ is either a program or a **TM**.
Reducing $X$ to CSAT

Thus an $\textbf{NP}$ problem is essentially a three tuple $\langle p, q, C \rangle$ where $C$ is either a program or $\textbf{TM}$.

**Problem X:** Given string $s$, is $s \in X$?

Same as the following: is there a proof $t$ of length $p(|s|)$ such that $C(s, t)$ says YES.

How do we reduce $X$ to CSAT? Need an algorithm $A$ that

1. takes $s$ (and $\langle p, q, C \rangle$) and creates a circuit $G$ in polynomial time in $|s|$ (note that $\langle p, q, C \rangle$ are fixed).
2. $G$ is satisfiable if and only if there is a proof $t$ such that $C(s, t)$ says YES.
Reducing $X$ to $\text{CSAT}$

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Thus an \textbf{NP} problem is essentially a three tuple \(\langle p, q, C \rangle\) where \(C\) is either a program or \textbf{TM}.

\textbf{Problem X}: Given string \(s\), is \(s \in X\)?

Same as the following: is there a proof \(t\) of length \(p(|s|)\) such that \(C(s, t)\) says YES.

How do we reduce \(X\) to \textbf{CSAT}? Need an algorithm \(A\) that

1. Takes \(s\) (and \(\langle p, q, C \rangle\)) and creates a circuit \(G\) in polynomial time in \(|s|\) (note that \(\langle p, q, C \rangle\) are fixed).

2. \(G\) is satisfiable if and only if there is a proof \(t\) such that \(C(s, t)\) says YES.
Reducing X to CSAT

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2. \( G \) is satisfiable if and only if there is a proof \( t \) such that \( C(s, t) \) says \text{YES}.

**Simple but Big Idea:** Programs are essentially the same as Circuits!

1. Convert \( C(s, t) \) into a circuit \( G \) with \( t \) as unknown inputs (rest is known including \( s \)).
2. We know that \(|t| = p(|s|)\) so express boolean string \( t \) as \( p(|s|) \) variables \( t_1, t_2, \ldots, t_k \) where \( k = p(|s|) \).
3. Asking if there is a proof \( t \) that makes \( C(s, t) \) say \text{YES} is same as whether there is an assignment of values to “unknown” variables \( t_1, t_2, \ldots, t_k \) that will make \( G \) evaluate to \text{true/YES}.
Reducing $X$ to CSAT

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Example: **Independent Set**

1. **Problem:** Does $G = (V, E)$ have an **Independent Set** of size $\geq k$?

2. **Certificate:** Set $S \subseteq V$.

3. **Certifier:** Check $|S| \geq k$ and no pair of vertices in $S$ is connected by an edge.

Formally, why is **Independent Set** in **NP**?
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Example: **Independent Set**

Formally why is **Independent Set** in **NP**?

1. **Input:** <
   \[ n, y_{1,1}, y_{1,2}, \ldots, y_{1,n}, y_{2,1}, \ldots, y_{2,n}, \ldots, y_{n,1}, \ldots, y_{n,n}, k > \]
   encodes < \[ G, k \] >.
   
   1. \( n \) is number of vertices in \( G \)
   2. \( y_{i,j} \) is a bit which is 1 if edge \((i, j)\) is in \( G \) and 0 otherwise (adjacency matrix representation)
   3. \( k \) is size of independent set.

2. **Certificate:** \( t = t_1 t_2 \ldots t_n \). Interpretation is that \( t_i \) is 1 if vertex \( i \) is in the independent set, 0 otherwise.
Certifier for Independent Set

Certifier $C(s, t)$ for Independent Set:

\[
\begin{align*}
\text{if } (t_1 + t_2 + \ldots + t_n < k) & \text{ then} \\
\quad & \text{return } NO \\
\text{else} & \\
\quad & \text{for each } (i, j) \text{ do} \\
\quad & \quad \text{if } (t_i \land t_j \land y_{i,j}) \text{ then} \\
\quad & \quad \quad \text{return } NO \\
\text{return } YES
\end{align*}
\]
Example: Independent Set

A certifier circuit for Independent Set

Figure: Graph $G$ with $k = 2$
Consider “program” $A$ that takes $f(|s|)$ steps on input string $s$.

**Question:** What computer is the program running on and what does *step* mean?

Real computers difficult to reason with mathematically because
- instruction set is too rich
- pointers and control flow jumps in one step
- assumption that pointer to code fits in one word

**Turing Machines**
- simpler model of computation to reason with
- can simulate real computers with *polynomial* slow down
- all moves are *local* (head moves only one cell)
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**Turing Machines**

1. simpler model of computation to reason with
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Certifiers that at TMs

Assume $C(\cdot, \cdot)$ is a (deterministic) Turing Machine $M$

Problem: Given $M$, input $s$, $p$, $q$ decide if there is a proof $t$ of length $p(|s|)$ such that $M$ on $s$, $t$ will halt in $q(|s|)$ time and say YES.

There is an algorithm $A$ that can reduce above problem to CSAT mechanically as follows.

1. $A$ first computes $p(|s|)$ and $q(|s|)$.
2. Knows that $M$ can use at most $q(|s|)$ memory/tape cells
3. Knows that $M$ can run for at most $q(|s|)$ time
4. Simulates the evolution of the state of $M$ and memory over time using a big circuit.
1 Think of $M$’s state at time $\ell$ as a string $x^\ell = x_1 x_2 \ldots x_k$ where each $x_i \in \{0, 1, B\} \times Q \cup \{q_{-1}\}$.

2 At time 0 the state of $M$ consists of input string $s$ a guess $t$ (unknown variables) of length $p(|s|)$ and rest $q(|s|)$ blank symbols.

3 At time $q(|s|)$ we wish to know if $M$ stops in $q_{\text{accept}}$ with say all blanks on the tape.

4 We write a circuit $C_\ell$ which captures the transition of $M$ from time $\ell$ to time $\ell + 1$.

5 Composition of the circuits for all times 0 to $q(|s|)$ gives a big (still poly) sized circuit $C$.

6 The final output of $C$ should be true if and only if the entire state of $M$ at the end leads to an accept state.
Key Ideas in reduction:

1. Use **TM**s as the code for certifier for simplicity
2. Since \( p() \) and \( q() \) are known to \( A \), it can set up all required memory and time steps in advance
3. Simulate computation of the **TM** from one time to the next as a circuit that only looks at three adjacent cells at a time

Note: Above reduction can be done to **SAT** as well. Reduction to **SAT** was the original proof of Steve Cook.
NP-Hardness of Circuit Satisfaction

Key Ideas in reduction:

1. Use \textbf{TM}s as the code for certifier for simplicity

2. Since \( p() \) and \( q() \) are known to \( \mathcal{A} \), it can set up all required memory and time steps in advance

3. Simulate computation of the \textbf{TM} from one time to the next as a circuit that only looks at three adjacent cells at a time

Note: Above reduction can be done to \textbf{SAT} as well. Reduction to \textbf{SAT} was the original proof of Steve Cook.
SAT is NP-Complete

1. We have seen that $\text{SAT} \in \text{NP}$

2. To show **NP-Hardness**, we will reduce Circuit Satisfiability (CSAT) to SAT

Instance of CSAT (we label each node):

![Diagram of a circuit satisfiability problem with labeled inputs and outputs.]

Input: $1, a, ?b, ?c, 0d, ?e$

Output: $\land, k$, $\neg, i$, $\land, j$, $\land, f$, $\lor, g$, $\lor, h$
Converting a circuit into a **CNF** formula

Label the nodes

(A) Input circuit

(B) Label the nodes.
Converting a circuit into a **CNF** formula

Introduce a variable for each node

(B) Label the nodes.  
(C) Introduce var for each node.
Converting a circuit into a \textbf{CNF} formula

Write a sub-formula for each variable that is true if the var is computed correctly.

(C) Introduce var for each node.

(D) Write a sub-formula for each variable that is true if the var is computed correctly.

\begin{align*}
x_k &= x_i \land x_k \quad \text{(Demand a sat' assignment!)} \\
x_j &= x_g \land x_h \\
x_i &= \neg x_f \\
x_h &= x_d \lor x_e \\
x_g &= x_b \lor x_c \\
x_f &= x_a \land x_b \\
x_d &= 0 \\
x_a &= 1
\end{align*}
Converting a circuit into a **CNF** formula

Convert each sub-formula to an equivalent CNF formula

<table>
<thead>
<tr>
<th>$x_k$</th>
<th>$x_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_k = x_i \land x_j$</td>
<td>$(\neg x_k \lor x_i) \land (\neg x_k \lor x_j) \land (x_k \lor \neg x_i \lor \neg x_j)$</td>
</tr>
<tr>
<td>$x_j = x_g \land x_h$</td>
<td>$(\neg x_j \lor x_g) \land (\neg x_j \lor x_h) \land (x_j \lor \neg x_g \lor \neg x_h)$</td>
</tr>
<tr>
<td>$x_i = \neg x_f$</td>
<td>$(x_i \lor x_f) \land (\neg x_i \lor \neg x_f)$</td>
</tr>
<tr>
<td>$x_h = x_d \lor x_e$</td>
<td>$(x_h \lor \neg x_d) \land (x_h \lor \neg x_e) \land (\neg x_h \lor x_d \lor x_e)$</td>
</tr>
<tr>
<td>$x_g = x_b \lor x_c$</td>
<td>$(x_g \lor \neg x_b) \land (x_g \lor \neg x_c) \land (\neg x_g \lor x_b \lor x_c)$</td>
</tr>
<tr>
<td>$x_f = x_a \land x_b$</td>
<td>$(\neg x_f \lor x_a) \land (\neg x_f \lor x_b) \land (x_f \lor \neg x_a \lor \neg x_b)$</td>
</tr>
<tr>
<td>$x_d = 0$</td>
<td>$\neg x_d$</td>
</tr>
<tr>
<td>$x_a = 1$</td>
<td>$x_a$</td>
</tr>
</tbody>
</table>
Converting a circuit into a **CNF** formula

Take the conjunction of all the **CNF** sub-formulas

\[ x_k \land (\neg x_k \lor x_i) \land (\neg x_k \lor x_j) \]
\[ \land (x_k \lor \neg x_i \lor \neg x_j) \land (\neg x_j \lor x_g) \]
\[ \land (\neg x_j \lor x_h) \land (x_j \lor \neg x_g \lor \neg x_h) \]
\[ \land (x_i \lor x_f) \land (\neg x_i \lor \neg x_f) \]
\[ \land (x_h \lor \neg x_d) \land (x_h \lor \neg x_e) \]
\[ \land (\neg x_h \lor x_d \lor x_e) \land (x_g \lor \neg x_b) \]
\[ \land (x_g \lor \neg x_c) \land (\neg x_g \lor x_b \lor x_c) \]
\[ \land (\neg x_f \lor x_a) \land (\neg x_f \lor x_b) \]
\[ \land (x_f \lor \neg x_a \lor \neg x_b) \land (\neg x_d) \land x_a \]

We got a **CNF** formula that is satisfiable if and only if the original circuit is satisfiable.
Reduction: \( \text{CSAT} \leq \text{P SAT} \)

1. For each gate (vertex) \( v \) in the circuit, create a variable \( x_v \).

2. Case \( \neg \): \( v \) is labeled \( \neg \) and has one incoming edge from \( u \) (so \( x_v = \neg x_u \)). In SAT formula generate, add clauses \( (x_u \lor x_v) \), \( (\neg x_u \lor \neg x_v) \). Observe that

\[
\begin{align*}
x_v = \neg x_u & \text{ is true } \iff \quad (x_u \lor x_v) \quad \text{both true.}
\end{align*}
\]
Reduction: \( \text{CSAT} \leq_p \text{SAT} \)

Continued...

\[ \text{Case } \lor: \text{ So } x_v = x_u \lor x_w. \text{ In } \text{SAT} \text{ formula generated, add clauses } (x_v \lor \neg x_u), (x_v \lor \neg x_w), \text{ and } (\neg x_v \lor x_u \lor x_w). \]

Again, observe that

\[
(x_v = x_u \lor x_w) \text{ is true } \iff (x_v \lor \neg x_u), (x_v \lor \neg x_w), \text{ all true.}
\]
Case $\land$: So $x_v = x_u \land x_w$. In SAT formula generated, add clauses $(\neg x_v \lor x_u)$, $(\neg x_v \lor x_w)$, and $(x_v \lor \neg x_u \lor \neg x_w)$. Again observe that

$$x_v = x_u \land x_w \text{ is true } \iff (\neg x_v \lor x_u), (\neg x_v \lor x_w), (x_v \lor \neg x_u \lor \neg x_w) \text{ all true.}$$
Reduction: \( \text{CSAT} \leq_{P} \text{SAT} \)

Continued...

1. If \( v \) is an input gate with a fixed value then we do the following. If \( x_v = 1 \) add clause \( x_v \). If \( x_v = 0 \) add clause \( \neg x_v \).

2. Add the clause \( x_v \) where \( v \) is the variable for the output gate.
Correctness of Reduction

Need to show circuit $C$ is satisfiable iff $\varphi_C$ is satisfiable

⇒ Consider a satisfying assignment $a$ for $C$
   1. Find values of all gates in $C$ under $a$
   2. Give value of gate $v$ to variable $x_v$; call this assignment $a'$
   3. $a'$ satisfies $\varphi_C$ (exercise)

⇐ Consider a satisfying assignment $a$ for $\varphi_C$
   1. Let $a'$ be the restriction of $a$ to only the input variables
   2. Value of gate $v$ under $a'$ is the same as value of $x_v$ in $a$
   3. Thus, $a'$ satisfies $C$

*Theorem*

*SAT* is NP-Complete.
Proving that a problem $X$ is **NP-Complete**

To prove $X$ is **NP-Complete**, show

1. **Show $X$ is in NP.**
   - certificate/proof of polynomial size in input
   - polynomial time certifier $C(s, t)$

2. **Reduction from a known **NP-Complete** problem such as CSAT or SAT to $X$**

$\text{SAT} \leq_P X$ implies that every **NP** problem $Y \leq_P X$. Why?

Transitivity of reductions:

$Y \leq_P \text{SAT}$ and $\text{SAT} \leq_P X$ and hence $Y \leq_P X$. 
To prove \( X \) is \textbf{NP-Complete}, show

1. Show \( X \) is in \textbf{NP}.
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2. Reduction from a known \textbf{NP-Complete} problem such as \textbf{CSAT} or \textbf{SAT} to \( X \)

\( \text{SAT} \leq_p X \) implies that every \textbf{NP} problem \( Y \leq_p X \). Why?

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\( Y \leq_p SAT \) and \( SAT \leq_p X \) and hence \( Y \leq_p X \).
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NP-Completeness via Reductions

1. CSAT is NP-Complete.
2. CSAT \leq_P SAT and SAT is in NP and hence SAT is NP-Complete.
3. SAT \leq_P 3-SAT and hence 3-SAT is NP-Complete.
4. 3-SAT \leq_P Independent Set (which is in NP) and hence Independent Set is NP-Complete.
5. Vertex Cover is NP-Complete.
6. Clique is NP-Complete.

Hundreds and thousands of different problems from many areas of science and engineering have been shown to be NP-Complete.

A surprisingly frequent phenomenon!
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