# CS 573: Algorithms

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University of Illinois, Urbana-Champaign

Fall 2013

# Administrivia, Introduction

Lecture 1

August 27, 2013

# The word "algorithm" comes from...

Muhammad ibn Musa al-Khwarizmi
780-850 AD
The word "algebra" is taken from the title of one of his books.

# Part I

# Administrivia

# Instructional Staff

1. Instructor:
   - Sariel Har-Peled (`sariel`)
2. Teaching Assistants:
   1. Ben Raichel (`raichel2`)
   2. David Holcomb (`dholcom2`)
3. Office hours: See course webpage
4. Email: See course webpage

# Online resources

1. Webpage: `courses.engr.illinois.edu/cs573/fa2013/`
   General information, homeworks, etc.
2. Moodle: Quizzes, solutions to homeworks.
3. Online questions/announcements: Piazza
   Online discussions, etc.

# Textbooks

1. Prerequisites: CS 173 (discrete math), CS 225 (data structures) and CS 373 (theory of computation)
2. Recommended books:
   1. Algorithms by Dasgupta, Papadimitriou & Vazirani. Available online for free!
   2. Algorithm Design by Kleinberg & Tardos
3. Lecture notes: Available on the web-page before/during/after every class.
4. Additional References
   1. Previous class notes of Jeff Erickson, Sariel Har-Peled and the instructor.
   2. Introduction to Algorithms: Cormen, Leiserson, Rivest, Stein.
   3. Computers and Intractability: Garey and Johnson.

# Prerequisites

1. Asymptotic notation: $O(), \Omega(), o()$.
2. Discrete Structures: sets, functions, relations, equivalence classes, partial orders, trees, graphs
3. Logic: predicate logic, boolean algebra
4. Proofs: **by induction**, by contradiction
5. Basic sums and recurrences: sum of a geometric series, unrolling of recurrences, basic calculus
6. Data Structures: arrays, multi-dimensional arrays, linked lists, trees, balanced search trees, heaps
7. Abstract Data Types: lists, stacks, queues, dictionaries, priority queues
8. Algorithms: sorting (merge, quick, insertion), pre/post/in order traversal of trees, depth/breadth first search of trees (maybe graphs)
9. Basic analysis of algorithms: loops and nested loops, deriving recurrences from a recursive program
10. Concepts from Theory of Computation: languages, automata, Turing machine, undecidability, non-determinism
11. Programming: in some general purpose language
12. Elementary Discrete Probability: event, random variable, independence
13. Mathematical maturity

# Grading Policy: Overview

1. Attendance/clickers: 5%
2. Quizzes: 5%
3. Homeworks: 15%
4. Midterm: 30%
5. Finals: 45% (covers the full course content)

# Homeworks

1. One quiz every 1-2-3 weeks: Due by midnight on Sunday.
2. One homework every 1-2-3 weeks.
3. Homeworks can be worked on in groups of up to 3 and each group submits *one* written solution (except Homework 0).
    1. Short quiz-style questions to be answered individually on *Moodle*.
4. Groups can be changed a *few* times only.

# More on Homeworks

1. No extensions or late homeworks accepted.
2. To compensate, the homework with the least score will be dropped in calculating the homework average.
3. Important: Read homework faq/instructions on website.

# Advice

1. Attend lectures, please ask plenty of questions.
2. Clickers...
3. Attend discussion sessions.
4. Don't skip homework and don't copy homework solutions.
5. Study regularly and keep up with the course.
6. Ask for help promptly. Make use of office hours.

# Homeworks

1. HW 0 is posted on the class website. Quiz 0 available
2. HW 0 to be submitted in individually.

# Part II

# Course Goals and Overview

# Topics

1. Some fundamental algorithms
2. Broadly applicable techniques in algorithm design
    1. Understanding problem structure
    2. Brute force enumeration and backtrack search
    3. Reductions
    4. Recursion
        1. Divide and Conquer
        2. Dynamic Programming
    5. Greedy methods
    6. Network Flows and Linear/Integer Programming (optional)
3. Analysis techniques
    1. Correctness of algorithms via induction and other methods
    2. Recurrences
    3. Amortization and elementary potential functions
4. Polynomial-time Reductions, NP-Completeness, Heuristics

# Goals

1. Algorithmic thinking
2. Learn/remember some basic tricks, algorithms, problems, ideas
3. Understand/appreciate limits of computation (intractability)
4. Appreciate the importance of algorithms in computer science and beyond (engineering, mathematics, natural sciences, social sciences, ...)
5. Have fun!!!

# Part III

# Algorithms and efficiency

# Primality testing

## Problem

Given an integer $N > 0$, is $N$ a prime?

**SimpleAlgorithm**:

**for** $i = 2$ to $\lfloor \sqrt{N} \rfloor$ **do**
  **if** $i$ divides $N$ **then**
    return ''COMPOSITE''
**return** ''PRIME''

Correctness? If $N$ is composite, at least one factor in $\{2, \ldots, \sqrt{N}\}$
Running time? $O(\sqrt{N})$ divisions? Sub-linear in input size! Wrong!

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# Primality testing

## ...Polynomial means... in input size

How many bits to represent $N$ in binary? $\lceil \log N \rceil$ bits.

Simple Algorithm takes $\sqrt{N} = 2^{(\log N)/2}$ time.
*Exponential* in the input size $n = \log N$.

1. Modern cryptography: binary numbers with 128, 256, 512 bits.
2. Simple Algorithm will take $2^{64}$, $2^{128}$, $2^{256}$ steps!
3. Fastest computer today about 3 petaFlops/sec: $3 \times 2^{50}$ floating point ops/sec.

**Lesson:**

Pay attention to representation size in analyzing efficiency of algorithms. Especially in *number* problems.

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# Efficient algorithms

So, is there an *efficient/good/effective* algorithm for primality?

**Question:**

What does efficiency mean?

In this class *efficiency* is broadly equated to *polynomial time*. $O(n), O(n \log n), O(n^2), O(n^3), O(n^{100}), \ldots$ where $n$ is size of the input.

Why? Is $n^{100}$ really efficient/practical? Etc.

Short answer: polynomial time is a robust, mathematically sound way to define efficiency. Has been useful for several decades.

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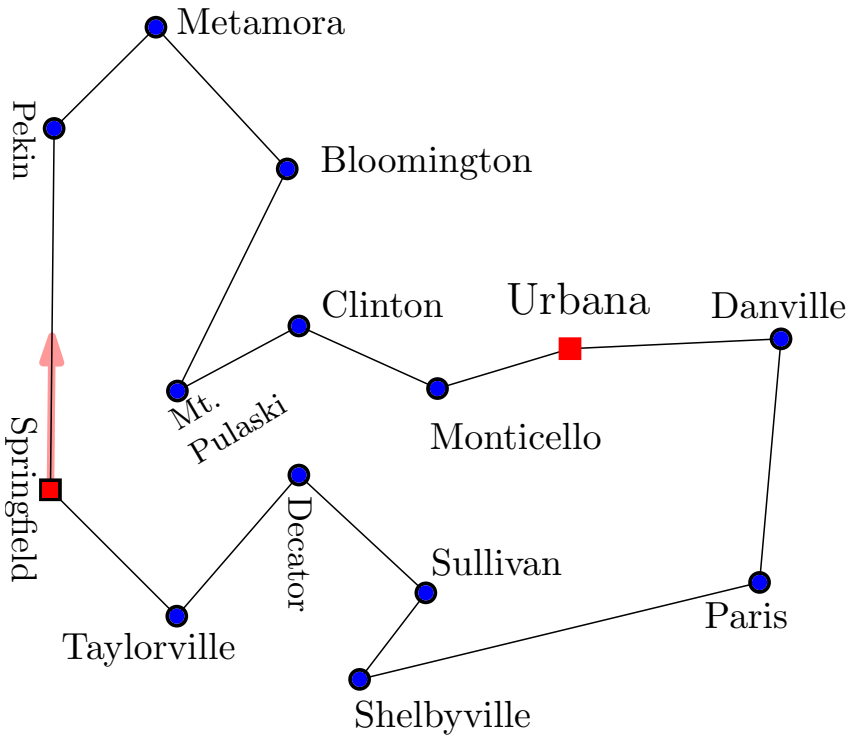
# TSP problem

## Lincoln's tour




1. Circuit court - ride through counties staying a few days in each town.
2. Lincoln was a lawyer traveling with the Eighth Judicial Circuit.
3. Picture: travel during 1850.
   1. Very close to optimal tour.
   2. Might have been optimal at the time..

# Solving TSP by a Computer

## Is it hard?

1. $n$ = number of cities.
2. $n^2$: size of input.
3. Number of possible solutions is

$$n * (n-1) * (n-2) * \ldots * 2 * 1 = n!.$$

4. $n!$ grows very quickly as $n$ grows.
   $n = 10$: $n! \approx 3628800$
   $n = 50$: $n! \approx 3 * 10^{64}$
   $n = 100$: $n! \approx 9 * 10^{157}$

# Solving TSP by a Computer

## Fastest computer...

1. Fastest super computer can do (roughly)

$$2.5 * 10^{15}$$

   operations a second.
2. Assume: computer checks $2.5 * 10^{15}$ solutions every second, then...
   1. $n = 20 \implies$ 2 hours.
   2. $n = 25 \implies$ 200 years.
   3. $n = 37 \implies 2 * 10^{20}$ years!!!

# What is a good algorithm?

## Running time...

| Input size | $n^2$ ops | $n^3$ ops | $n^4$ ops | $n!$ ops |
|---|---|---|---|---|
| 5 | 0 secs | 0 secs | 0 secs | 0 secs |
| 20 | 0 secs | 0 secs | 0 secs | 16 mins |
| 30 | 0 secs | 0 secs | 0 secs | $3 \cdot 10^9$ years |
| 100 | 0 secs | 0 secs | 0 secs | never |
| 8000 | 0 secs | 0 secs | 1 secs | never |
| 16000 | 0 secs | 0 secs | 26 secs | never |
| 32000 | 0 secs | 0 secs | 6 mins | never |
| 64000 | 0 secs | 0 secs | 111 mins | never |
| 200,000 | 0 secs | 3 secs | 7 days | never |
| 2,000,000 | 0 secs | 53 mins | 202.943 years | never |
| $10^8$ | 4 secs | 12.6839 years | $10^9$ years | never |
| $10^9$ | 6 mins | 12683.9 years | $10^{13}$ years | never |

# What is a good algorithm?

## Running time...

# Primes is in $\mathbf{P}$!

## Theorem (Agrawal-Kayal-Saxena'02)

*There is a polynomial time algorithm for primality.*

First polynomial time algorithm for testing primality. Running time is $O(\log^{12} N)$ further improved to about $O(\log^{6} N)$ by others. In terms of input size $n = \log N$, time is $O(n^6)$.

# What about before 2002?

Primality testing a key part of cryptography. What was the algorithm being used before 2002?
Miller-Rabin *randomized* algorithm:

1. runs in polynomial time: $O(\log^3 N)$ time
2. if $N$ is prime correctly says "yes".
3. if $N$ is composite it says "yes" with probability at most $1/2^{100}$ (can be reduced further at the expense of more running time).

Based on Fermat's little theorem and some basic number theory.

# Factoring

1. Modern public-key cryptography based on RSA (Rivest-Shamir-Adelman) system.
2. Relies on the difficulty of factoring a composite number into its prime factors.
3. There is a polynomial time algorithm that decides whether a given number $N$ is prime or not (hence composite or not) but no known polynomial time algorithm to factor a given number.

**Lesson**

Intractability can be useful!

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# Unit-Cost RAM Model

Informal description:

1. Basic data type is an integer/floating point number
2. Numbers in input fit in a word
3. Arithmetic/comparison operations on words take constant time
4. Arrays allow random access (constant time to access $A[i]$)
5. Pointer based data structures via storing addresses in a word

# Example

Sorting: input is an array of $n$ numbers

1. input size is $n$ (ignore the bits in each number),
2. comparing two numbers takes $O(1)$ time,
3. random access to array elements,
4. addition of indices takes constant time,
5. basic arithmetic operations take constant time,
6. reading/writing one word from/to memory takes constant time.

We will usually not allow (or be careful about allowing):

1. bitwise operations (and, or, xor, shift, etc).
2. floor function.
3. limit word size (usually assume unbounded word size).

# Caveats of RAM Model

Unit-Cost RAM model is applicable in wide variety of settings in practice. However it is not a proper model in several important situations so one has to be careful.

1. For some problems such as basic arithmetic computation, unit-cost model makes no sense. Examples: multiplication of two $n$-digit numbers, primality etc.
2. Input data is very large and does not satisfy the assumptions that individual numbers fit into a word or that total memory is bounded by $2^k$ where $k$ is word length.
3. Assumptions valid only for certain type of algorithms that do not create large numbers from initial data. For example, exponentiation creates very big numbers from initial numbers.

# Models used in class

In this course:

1. Assume unit-cost $\mathrm{RAM}$ by default.
2. We will explicitly point out where unit-cost RAM is not applicable for the problem at hand.

# Part IV

# Reductions

# Independent Sets and Cliques

Given a graph $G$, a set of vertices $V'$ is:

1. ***independent set***: no two vertices of $V'$ connected by an edge.

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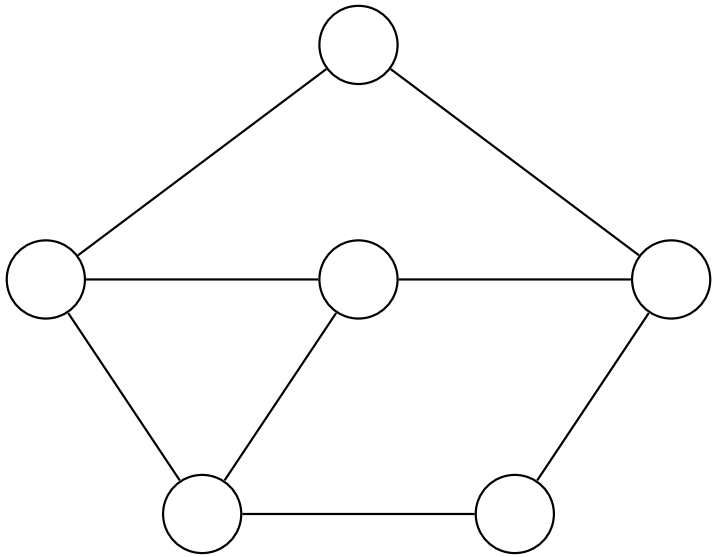

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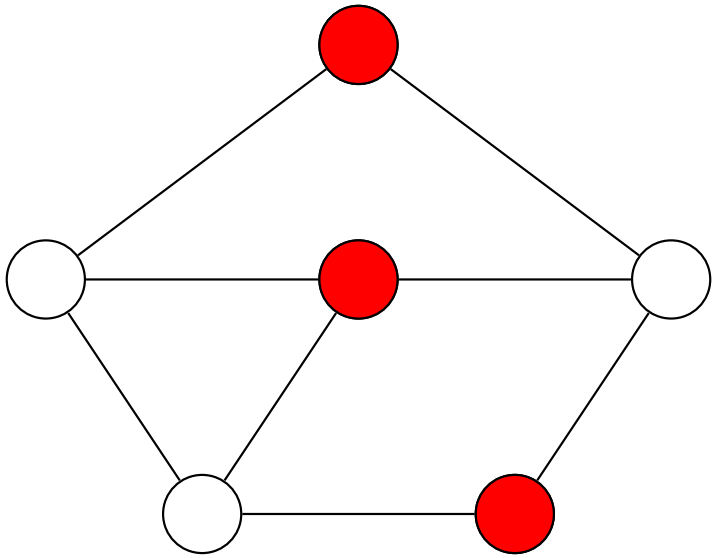

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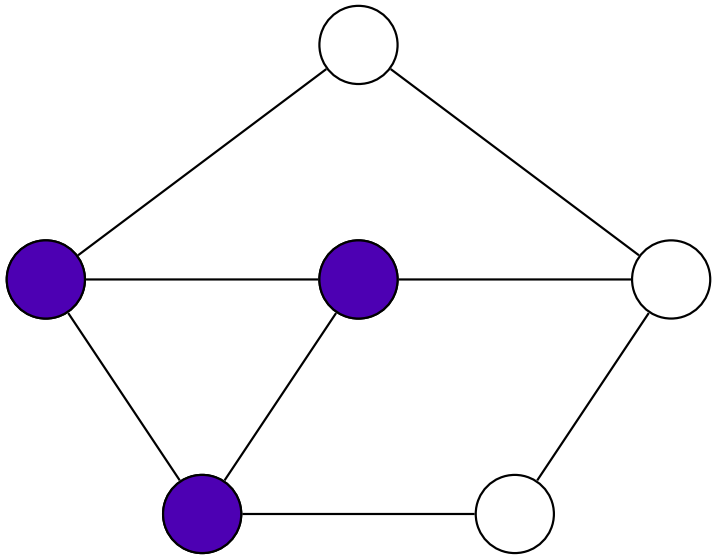

# The **Independent Set** and **Clique** Problems

## Independent Set

**Instance**: A graph **G** and an integer $k$.
**Question:** Does **G** has an independent set of size $\geq k$?

## Clique

**Instance**: A graph **G** and an integer $k$.
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# Types of Problems

## Decision, Search, and Optimization

1. ***Decision problem***. Example: given $n$, is $n$ prime?.
2. ***Search problem***. Example: given $n$, find a factor of $n$ if it exists.
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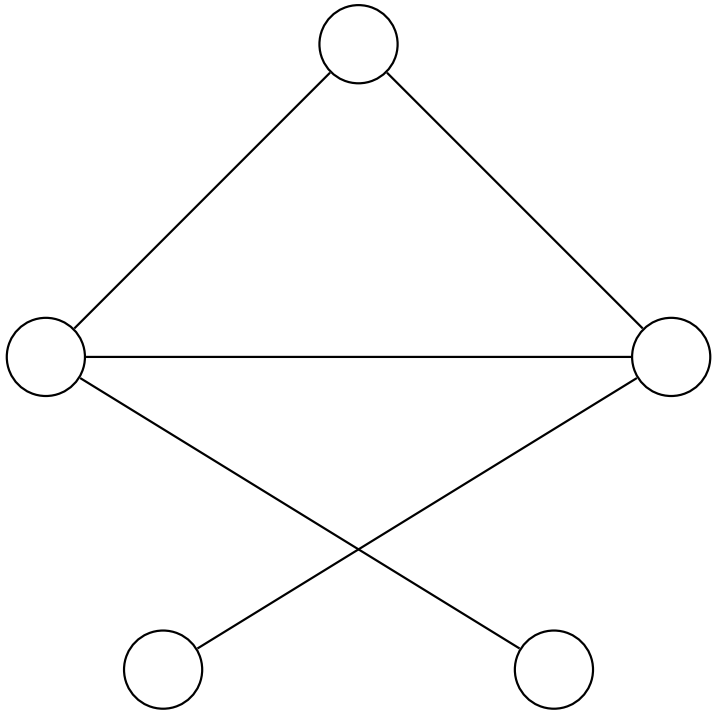

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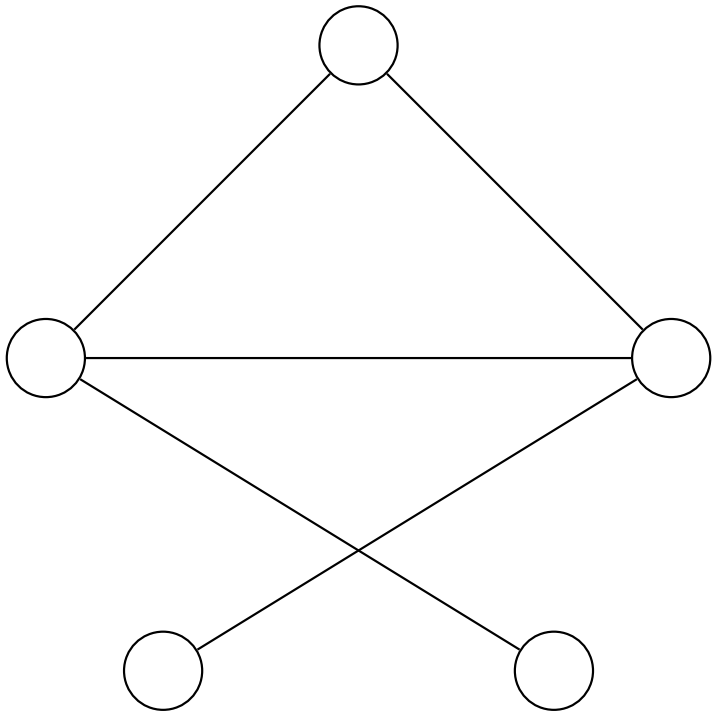
An instance of **Independent Set** is a graph $G$ and an integer $k$.

Convert $G$ to $\overline{G}$, in which $(u, v)$ is an edge iff $(u, v)$ is not an edge of $G$. ($\overline{G}$ is the *complement* of $G$.)
We use $\overline{G}$ and $k$ as the instance of **Clique**.



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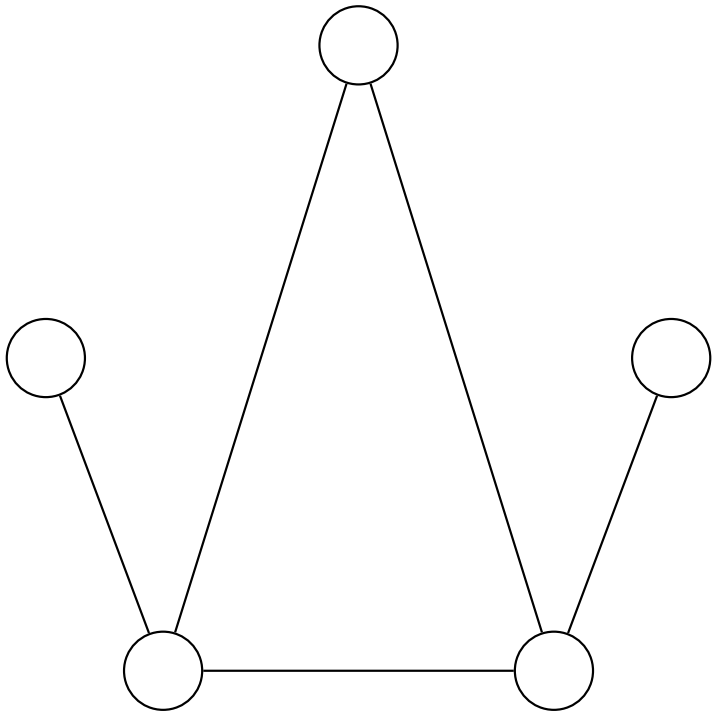

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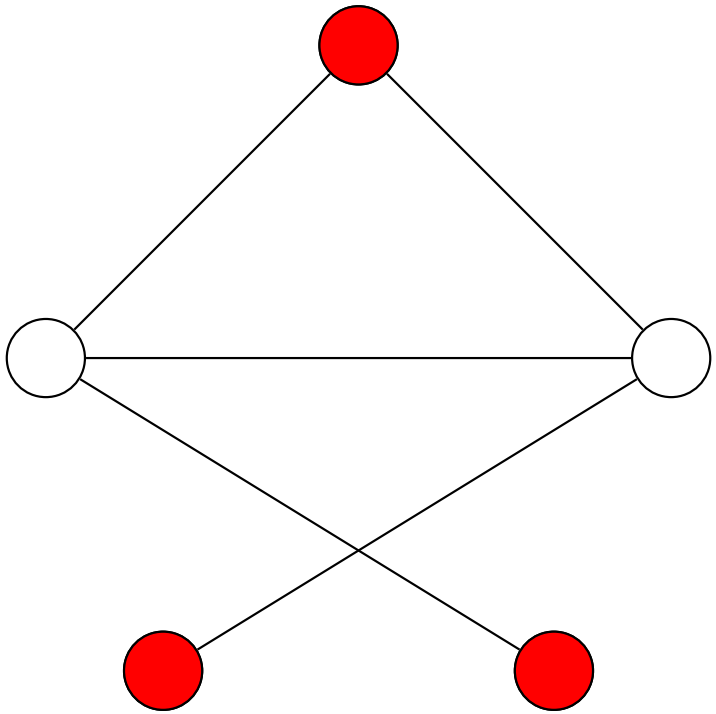

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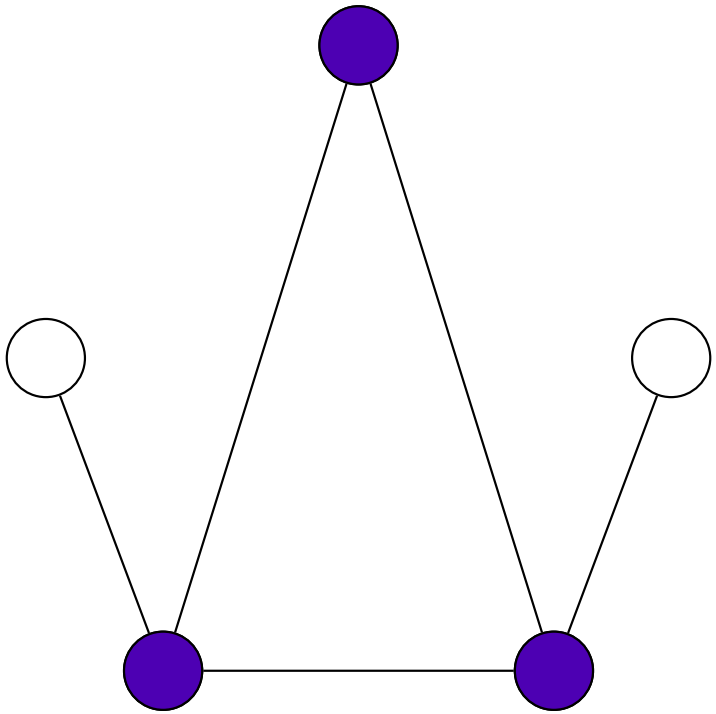

# Independent Set and Clique

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   What does this mean?
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# Using reductions to solve problems

1. $\mathcal{R}$: Reduction $X \rightarrow Y$
2. $\mathcal{A}_Y$: algorithm for $Y$:
3. $\Longrightarrow$ New algorithm for $X$:

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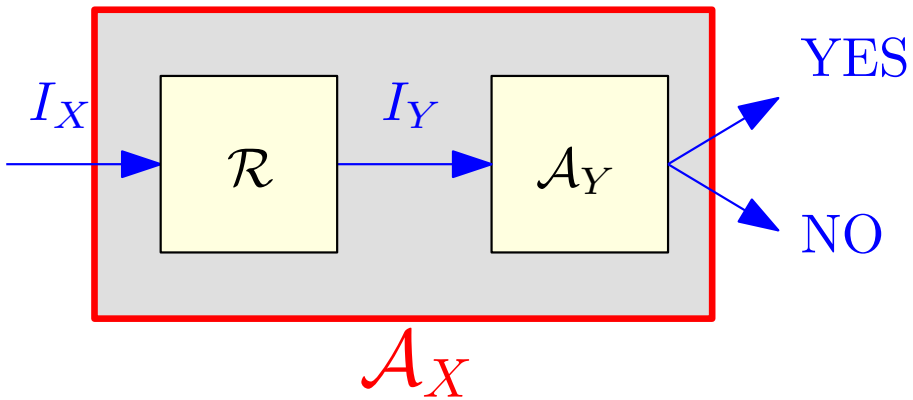
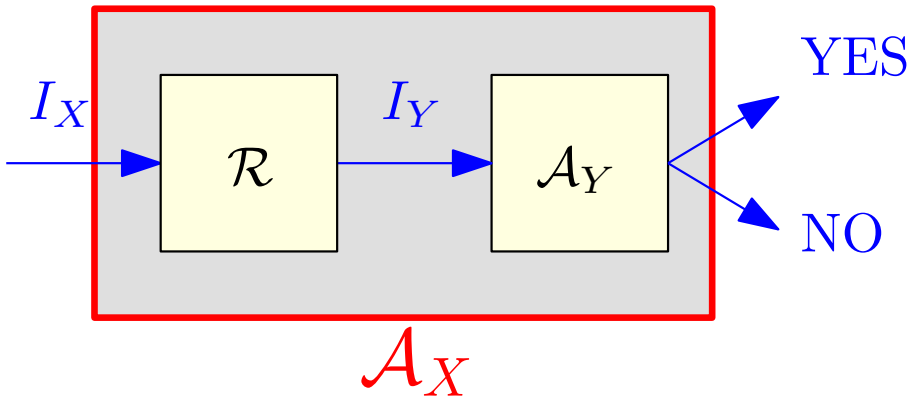



If $\mathcal{R}$ and $\mathcal{A}_Y$ polynomial-time $\Longrightarrow$ $\mathcal{A}_X$ polynomial-time.

# Comparing Problems

1. "Problem $X$ is no harder to solve than Problem $Y$".
2. If Problem $X$ reduces to Problem $Y$ (we write $X \leq Y$), then $X$ cannot be harder to solve than $Y$.
3. $X \leq Y$:
   1. $X$ is no harder than $Y$, or
   2. $Y$ is at least as hard as $X$.

# Polynomial-time reductions

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3. $X \leq_P Y$: Have polynomial-time reduction from problem $X$ to problem $Y$.
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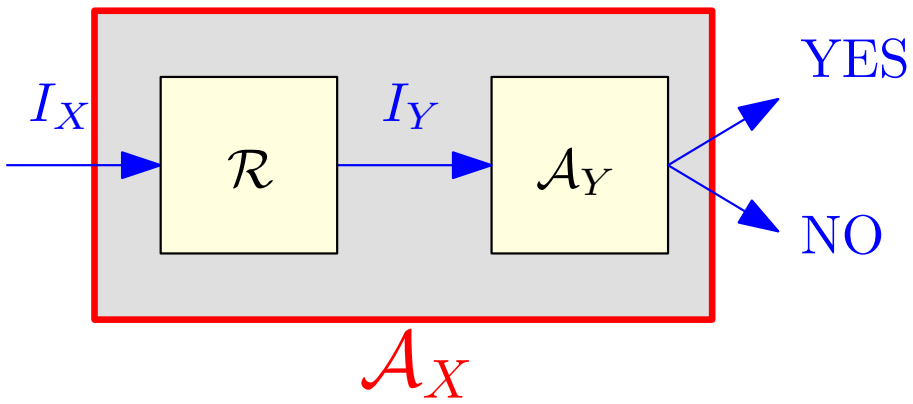


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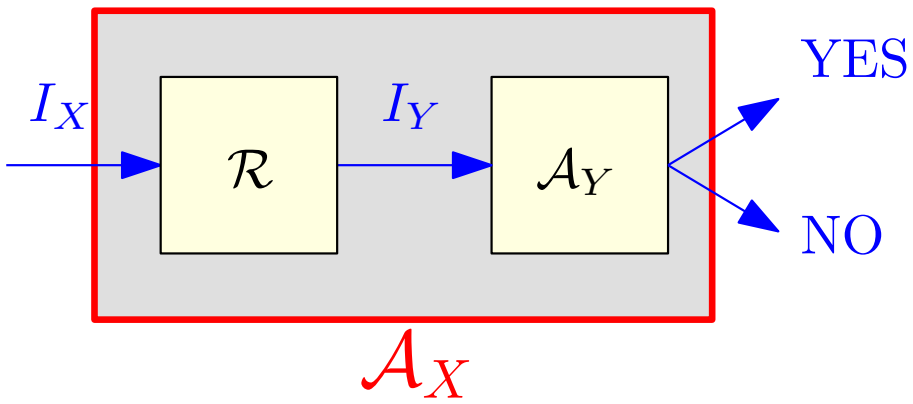


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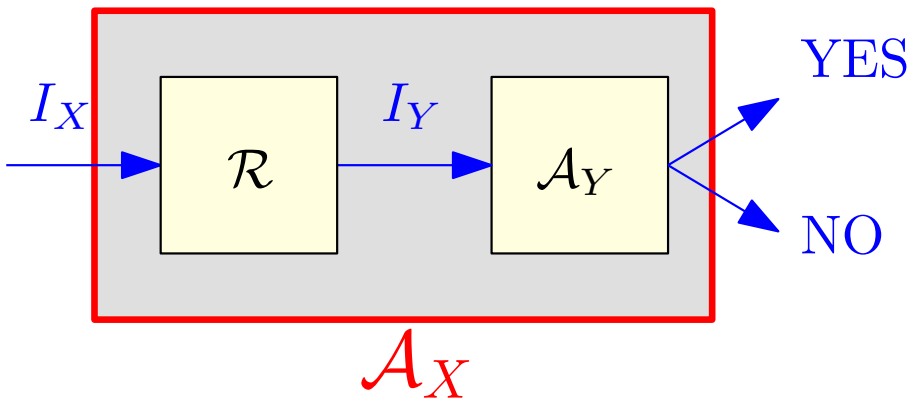


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# Polynomial-time reductions and hardness

## Lemma

*For decision problems $X$ and $Y$, if $X \leq_P Y$, and $Y$ has an efficient algorithm, $X$ has an efficient algorithm.*

1. **Independent Set**: "believe" there is no efficient algorithm.
2. What about **Clique**?
3. Showed: **Independent Set** $\leq_P$ **Clique**.
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## Observation

*If $X \leq_P Y$ and $X$ does not have an efficient algorithm, $Y$ cannot have an efficient algorithm!*

# Polynomial-time reductions and instance sizes

## Proposition

*$\mathcal{R}$: a polynomial-time reduction from $X$ to $Y$.*
*Then, for any instance $I_X$ of $X$, the size of the instance $I_Y$ of $Y$ produced from $I_X$ by $\mathcal{R}$ is polynomial in the size of $I_X$.*

## Proof.

$\mathcal{R}$ is a polynomial-time algorithm and hence on input $I_X$ of size $|I_X|$ it runs in time $p(|I_X|)$ for some polynomial $p()$.
$I_Y$ is the output of $\mathcal{R}$ on input $I_X$.
$\mathcal{R}$ can write at most $p(|I_X|)$ bits and hence $|I_Y| \leq p(|I_X|)$. □

Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.

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# Polynomial-time Reduction

## Definition

A ***polynomial time reduction*** from a *decision* problem $X$ to a *decision* problem $Y$ is an *algorithm* $\mathcal{A}$ such that:

1. Given an instance $I_X$ of $X$, $\mathcal{A}$ produces an instance $I_Y$ of $Y$.
2. $\mathcal{A}$ runs in time polynomial in $|I_X|$. This implies that $|I_Y|$ (size of $I_Y$) is polynomial in $|I_X|$.
3. Answer to $I_X$ YES *iff* answer to $I_Y$ is YES.

## Proposition

*If $X \leq_P Y$ then a polynomial time algorithm for $Y$ implies a polynomial time algorithm for $X$.*

This is a ***Karp reduction***.

# Transitivity of Reductions

## Proposition

*$X \leq_P Y$ and $Y \leq_P Z$ implies that $X \leq_P Z$.*

1. Note: $X \leq_P Y$ does not imply that $Y \leq_P X$ and hence it is very important to know the FROM and TO in a reduction.
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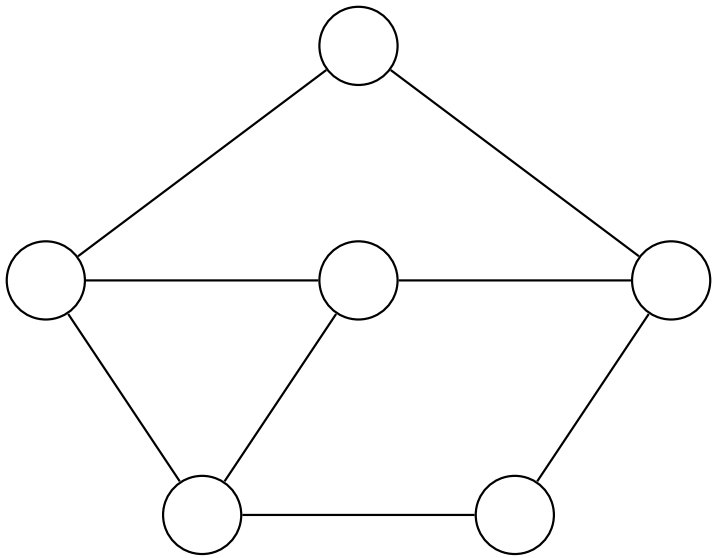

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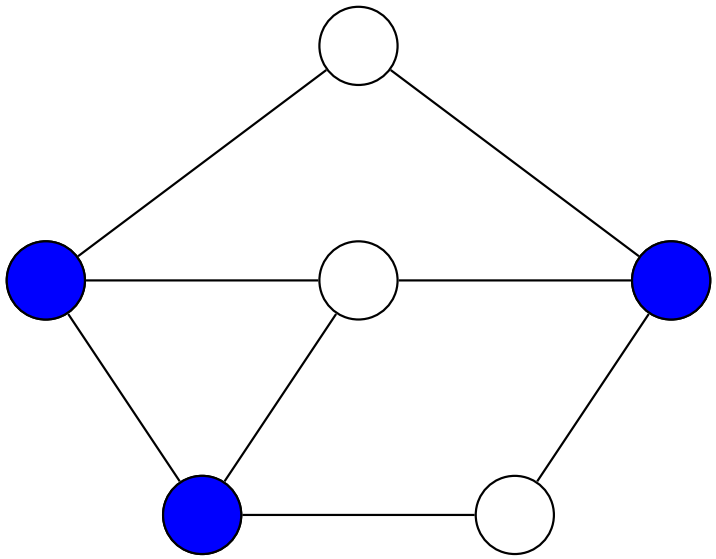

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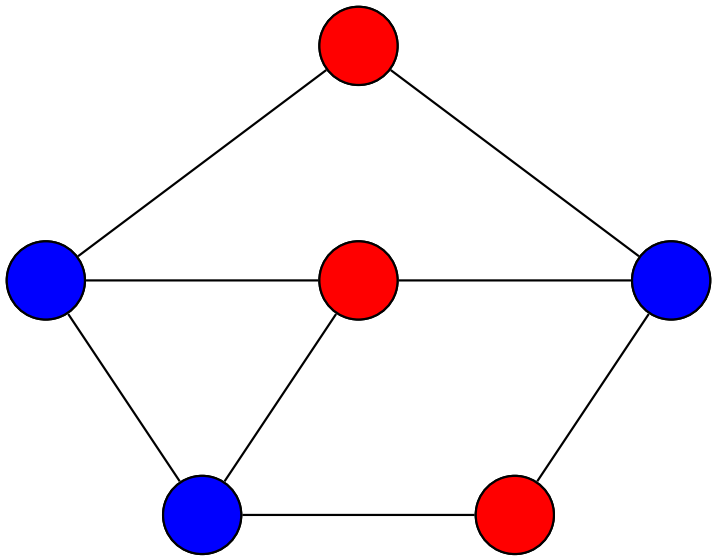

# The **Vertex Cover** Problem

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# Relationship between...

## Vertex Cover and Independent Set

### Proposition

*Let $G = (V, E)$ be a graph. $S$ is an independent set if and only if $V \setminus S$ is a vertex cover.*

### Proof.

($\Rightarrow$) Let $S$ be an independent set

1. Consider any edge $uv \in E$.
2. Since $S$ is an independent set, either $u \notin S$ or $v \notin S$.
3. Thus, either $u \in V \setminus S$ or $v \in V \setminus S$.
4. $V \setminus S$ is a vertex cover.

($\Leftarrow$) Let $V \setminus S$ be some vertex cover:

1. Consider $u, v \in S$
2. $uv$ is not an edge of $\mathbf{G}$, as otherwise $V \setminus S$ does not cover $uv$.
3. $\Longrightarrow$ $S$ is thus an independent set. □

# Independent Set $\leq_P$ Vertex Cover

1. $G$: graph with $n$ vertices, and an integer $k$ be an instance of the **Independent Set** problem.
2. $G$ has an independent set of size $\geq k$ iff $G$ has a vertex cover of size $\leq n - k$
3. $(G, k)$ is an instance of **Independent Set** , and $(G, n - k)$ is an instance of **Vertex Cover** with the same answer.
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## Problem (Set Cover)

**Input:** *Given a set $U$ of $n$ elements, a collection $S_1, S_2, \ldots S_m$ of subsets of $U$, and an integer $k$.*

**Goal:** *Is there a collection of at most $k$ of these sets $S_i$ whose union is equal to $U$?*

## Example

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $k = 2$ with

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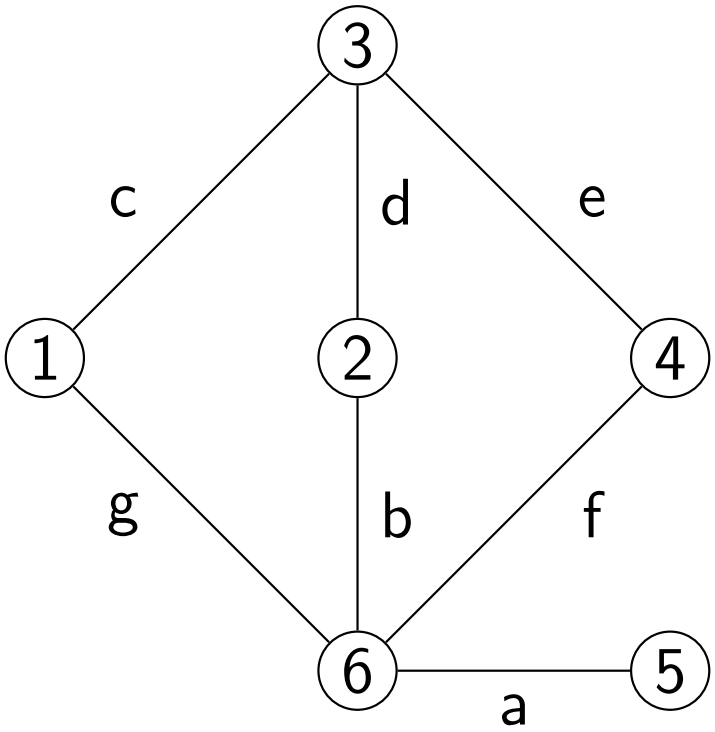
Observe that $G$ has vertex cover of size $k$ if and only if $U, \{S_v\}_{v \in V}$ has a set cover of size $k$. (Exercise: Prove this.)

# Vertex Cover $\leq_P$ Set Cover: Example

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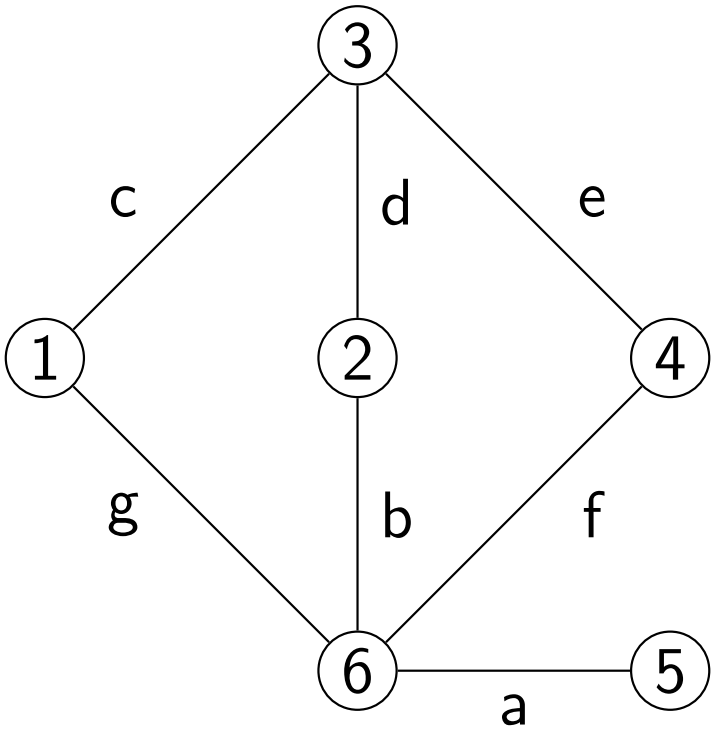


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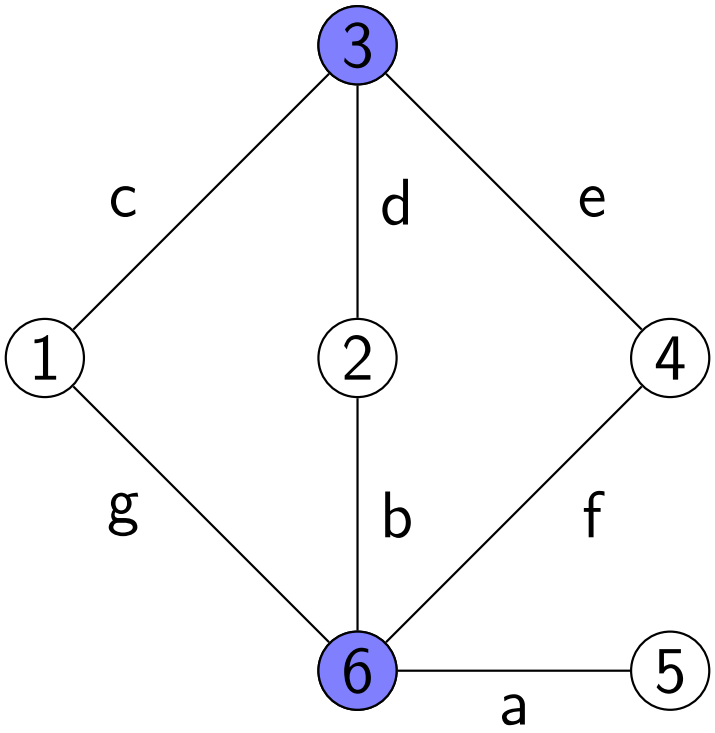


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# Proving Reductions

To prove that $X \leq_P Y$ you need to give an algorithm $\mathcal{A}$ that:

1. Transforms an instance $I_X$ of $X$ into an instance $I_Y$ of $Y$.
2. Satisfies the property that answer to $I_X$ is YES iff $I_Y$ is YES.
   1. typical easy direction to prove: answer to $I_Y$ is YES if answer to $I_X$ is YES
   2. typical difficult direction to prove: answer to $I_X$ is YES if answer to $I_Y$ is YES (equivalently answer to $I_X$ is NO if answer to $I_Y$ is NO).
3. Runs in ***polynomial*** time.

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We looked at polynomial-time reductions.

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1. If $X \leq_P Y$, and there is no efficient algorithm for $X$, there is no efficient algorithm for $Y$.

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We looked at some examples of reductions between **Independent Set**, **Clique**, **Vertex Cover**, and **Set Cover**.

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