

# Chapter 26

## Entropy, Randomness, and Information

By Sarel Har-Peled, December 7, 2009<sup>①</sup>

“If only once - only once - no matter where, no matter before what audience - I could better the record of the great Rastelli and juggle with thirteen balls, instead of my usual twelve, I would feel that I had truly accomplished something for my country. But I am not getting any younger, and although I am still at the peak of my powers there are moments - why deny it? - when I begin to doubt - and there is a time limit on all of us.”

– Romain Gary, The talent scout.

### 26.1 Entropy

**Definition 26.1.1** The *entropy* in bits of a discrete random variable  $X$  is given by

$$\mathbb{H}(X) = - \sum_x \Pr[X = x] \lg \Pr[X = x].$$

Equivalently,  $\mathbb{H}(X) = \mathbf{E}\left[\lg \frac{1}{\Pr[X]}\right]$ .

The *binary entropy* function  $\mathbb{H}(p)$  for a random binary variable that is 1 with probability  $p$ , is  $\mathbb{H}(p) = -p \lg p - (1-p) \lg(1-p)$ . We define  $\mathbb{H}(0) = \mathbb{H}(1) = 0$ .

The function  $\mathbb{H}(p)$  is a concave symmetric around  $1/2$  on the interval  $[0, 1]$  and achieves its maximum at  $1/2$ . For a concrete example, consider  $\mathbb{H}(3/4) \approx 0.8113$  and  $\mathbb{H}(7/8) \approx 0.5436$ . Namely, a coin that has  $3/4$  probably to be heads have higher amount of “randomness” in it than a coin that has probability  $7/8$  for heads.

We have  $\mathbb{H}'(p) = -\lg p + \lg(1-p) = \lg \frac{1-p}{p}$  and  $\mathbb{H}''(p) = \frac{p}{1-p} \cdot \left(-\frac{1}{p^2}\right) = -\frac{1}{p(1-p)}$ . Thus,  $\mathbb{H}''(p) \leq 0$ , for all  $p \in (0, 1)$ , and the  $\mathbb{H}(\cdot)$  is concave in this range. Also,  $\mathbb{H}'(1/2) = 0$ , which implies that  $\mathbb{H}(1/2) = 1$  is a maximum of the binary entropy. Namely, a balanced coin has the largest amount of randomness in it.

**Example 26.1.2** A random variable  $X$  that has probability  $1/n$  to be  $i$ , for  $i = 1, \dots, n$ , has entropy  $\mathbb{H}(X) = -\sum_{i=1}^n \frac{1}{n} \lg \frac{1}{n} = \lg n$ .

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Note, that the entropy is oblivious to the exact values that the random variable can have, and it is sensitive only to the probability distribution. Thus, a random variables that accepts  $-1, +1$  with equal probability has the same entropy (i.e., 1) as a fair coin.

**Lemma 26.1.3** *Let  $X$  and  $Y$  be two independent random variables, and let  $Z$  be the random variable  $(X, Y)$ . Then  $\mathbb{H}(Z) = \mathbb{H}(X) + \mathbb{H}(Y)$ .*

*Proof:* In the following, summation are over all possible values that the variables can have. By the independence of  $X$  and  $Y$  we have

$$\begin{aligned}
\mathbb{H}(Z) &= \sum_{x,y} \Pr[(X, Y) = (x, y)] \lg \frac{1}{\Pr[(X, Y) = (x, y)]} \\
&= \sum_{x,y} \Pr[X = x] \Pr[Y = y] \lg \frac{1}{\Pr[X = x] \Pr[Y = y]} \\
&= \sum_x \sum_y \Pr[X = x] \Pr[Y = y] \lg \frac{1}{\Pr[X = x]} \\
&\quad + \sum_y \sum_x \Pr[X = x] \Pr[Y = y] \lg \frac{1}{\Pr[Y = y]} \\
&= \sum_x \Pr[X = x] \lg \frac{1}{\Pr[X = x]} + \sum_y \Pr[Y = y] \lg \frac{1}{\Pr[Y = y]} = \mathbb{H}(X) + \mathbb{H}(Y).
\end{aligned}$$

**Lemma 26.1.4** *Suppose that  $nq$  is integer in the range  $[0, n]$ . Then  $\frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{nq} \leq 2^{n\mathbb{H}(q)}$ .*

*Proof:* This trivially holds if  $q = 0$  or  $q = 1$ , so assume  $0 < q < 1$ . We know that

$$\binom{n}{nq} q^{nq} (1-q)^{n-nq} \leq (q + (1-q))^n = 1.$$

As such, since  $q^{-nq} (1-q)^{-(1-q)n} = 2^{n(-q \lg q - (1-q) \lg(1-q))} = 2^{n\mathbb{H}(q)}$ , we have

$$\binom{n}{nq} \leq q^{-nq} (1-q)^{-(1-q)n} = 2^{n\mathbb{H}(q)}.$$

As for the other direction, let  $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$ . We claim that  $\mu(nq) = \binom{n}{nq} q^{nq} (1-q)^{n-nq}$  is the largest term in  $\sum_{k=0}^n \mu(k) = 1$ . Indeed,

$$\Delta_k = \mu(k) - \mu(k+1) = \binom{n}{k} q^k (1-q)^{n-k} \left( 1 - \frac{n-k}{k+1} \frac{q}{1-q} \right),$$

and the sign of this quantity is the sign of the last term, which is

$$\text{sign}(\Delta_k) = \text{sign} \left( 1 - \frac{(n-k)q}{(k+1)(1-q)} \right) = \text{sign} \left( \frac{(k+1)(1-q) - (n-k)q}{(k+1)(1-q)} \right).$$

Now,

$$(k+1)(1-q) - (n-k)q = k+1 - kq - q - nq + kq = 1 + k - q - nq.$$

Namely,  $\Delta_k \geq 0$  when  $k \geq nq + q - 1$ , and  $\Delta_k < 0$  otherwise. Namely,  $\mu(k) < \mu(k + 1)$ , for  $k < nq$ , and  $\mu(k) \geq \mu(k + 1)$  for  $k \geq nq$ . Namely,  $\mu(nq)$  is the largest term in  $\sum_{k=0}^n \mu(k) = 1$ , and as such it is larger than the average. We have  $\mu(nq) = \binom{n}{nq} q^{nq} (1 - q)^{n-nq} \geq \frac{1}{n+1}$ , which implies

$$\binom{n}{nq} \geq \frac{1}{n+1} q^{-nq} (1 - q)^{-(n-nq)} = \frac{1}{n+1} 2^{n\mathbb{H}(q)}.$$

Lemma 26.1.4 can be extended to handle non-integer values of  $q$ . This is straightforward, and we omit the easy details. ■

**Corollary 26.1.5** *We have:*

(i)  $q \in [0, 1/2] \Rightarrow \binom{n}{\lfloor nq \rfloor} \leq 2^{n\mathbb{H}(q)}$ .    (ii)  $q \in [1/2, 1] \Rightarrow \binom{n}{\lceil nq \rceil} \leq 2^{n\mathbb{H}(q)}$ .  
(iii)  $q \in [1/2, 1] \Rightarrow \frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{\lfloor nq \rfloor}$ .    (iv)  $q \in [0, 1/2] \Rightarrow \frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{\lceil nq \rceil}$ .

The bounds of Lemma 26.1.4 and Corollary 26.1.5 are loose but sufficient for our purposes. As a sanity check, consider the case when we generate a sequence of  $n$  bits using a coin with probability  $q$  for head, then by the Chernoff inequality, we will get roughly  $nq$  heads in this sequence. As such, the generated sequence  $Y$  belongs to  $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$  possible sequences that have similar probability. As such,  $\mathbb{H}(Y) \approx \lg \binom{n}{nq} = n\mathbb{H}(q)$ , by Example 26.1.2, a fact that we already know from Lemma 26.1.3.

## 26.1.1 Extracting randomness

Entropy can be interpreted as the amount of unbiased random coin flips can be extracted from a random variable.

**Definition 26.1.6** An extraction function **Ext** takes as input the value of a random variable  $X$  and outputs a sequence of bits  $y$ , such that  $\Pr[\mathbf{Ext}(X) = y \mid |y| = k] = \frac{1}{2^k}$ , whenever  $\Pr[|y| = k] > 0$ , where  $|y|$  denotes the length of  $y$ .

As a concrete (easy) example, consider  $X$  to be a uniform random integer variable out of  $0, \dots, 7$ . All that **Ext**( $X$ ) has to do in this case, is to compute the binary representation of  $x$ . However, note that Definition 26.1.6 is somewhat more subtle, as it requires that all extracted sequence of the same length would have the same probability.

Thus, for  $X$  a uniform random integer variable in the range  $0, \dots, 11$ , the function **Ext**( $x$ ) can output the binary representation for  $x$  if  $0 \leq x \leq 7$ . However, what do we do if  $x$  is between 8 and 11? The idea is to output the binary representation of  $x - 8$  as a two bit number. Clearly, Definition 26.1.6 holds for this extraction function, since  $\Pr[\mathbf{Ext}(X) = 00 \mid |\mathbf{Ext}(X)| = 2] = \frac{1}{4}$ , as required. This scheme can be of course extracted for any range.

The following is obvious, but we provide a proof anyway.

**Lemma 26.1.7** *Let  $x/y$  be a fraction, such that  $x/y < 1$ . Then, for any  $i$ , we have  $x/y < (x+i)/(y+i)$ .*

*Proof:* We need to prove that  $x(y+i) - (x+i)y < 0$ . The left side is equal to  $i(x-y)$ , but since  $y > x$  (as  $x/y < 1$ ), this quantity is negative, as required. ■

**Theorem 26.1.8** *Suppose that the value of a random variable  $X$  is chosen uniformly at random from the integers  $\{0, \dots, m-1\}$ . Then there is an extraction function for  $X$  that outputs on average at least  $\lfloor \lg m \rfloor - 1 = \lfloor \mathbb{H}(X) \rfloor - 1$  independent and unbiased bits.*

*Proof:* We represent  $m$  as a sum of unique powers of 2, namely  $m = \sum_i a_i 2^i$ , where  $a_i \in \{0, 1\}$ . Thus, we decomposed  $\{0, \dots, m-1\}$  into a disjoint union of blocks that have sizes which are distinct powers of 2. If a number falls inside such a block, we output its relative location in the block, using binary representation of the appropriate length (i.e.,  $k$  if the block is of size  $2^k$ ). The fact that this is an extraction function, fulfilling Definition 26.1.6, is obvious.

Now, observe that the claim holds trivially if  $m$  is a power of two. Thus, consider the case that  $m$  is not a power of 2. If  $X$  falls inside a block of size  $2^k$  then the entropy is  $k$ . Thus, for the inductive proof, assume that are looking at the largest block in the decomposition, that is  $m < 2^{k+1}$ , and let  $u = \lfloor \lg(m - 2^k) \rfloor < k$ . There must be a block of size  $2^u$  in the decomposition of  $m$ . Namely, we have two blocks that we know in the decomposition of  $m$ , of sizes  $2^k$  and  $2^u$ . Note, that these two blocks are the largest blocks in the decomposition of  $m$ . In particular,  $2^k + 2 * 2^u > m$ , implying that  $2^{u+1} + 2^k - m > 0$ .

Let  $Y$  be the random variable which is the number of bits output by the extractor algorithm.

By Lemma 26.1.7, since  $\frac{m-2^k}{m} < 1$ , we have

$$\frac{m-2^k}{m} \leq \frac{m-2^k + (2^{u+1} + 2^k - m)}{m + (2^{u+1} + 2^k - m)} = \frac{2^{u+1}}{2^{u+1} + 2^k}.$$

Thus, by induction (we assume the claim holds for all integers smaller than  $m$ ), we have

$$\begin{aligned} \mathbf{E}[Y] &\geq \frac{2^k}{m}k + \frac{m-2^k}{m} \left( \underbrace{\lfloor \lg(m-2^k) \rfloor}_u - 1 \right) = \frac{2^k}{m}k + \frac{m-2^k}{m} \underbrace{(k-k)}_{=0} + u - 1 = k + \frac{m-2^k}{m}(u-k-1) \\ &\geq k + \frac{2^{u+1}}{2^{u+1} + 2^k} (u-k-1) = k - \frac{2^{u+1}}{2^{u+1} + 2^k} (1+k-u), \end{aligned}$$

since  $u-k-1 \leq 0$  as  $k > u$ . If  $u = k-1$ , then  $\mathbf{E}[Y] \geq k - \frac{1}{2} \cdot 2 = k-1$ , as required. If  $u = k-2$  then  $\mathbf{E}[Y] \geq k - \frac{1}{3} \cdot 3 = k-1$ . Finally, if  $u < k-2$  then

$$\mathbf{E}[Y] \geq k - \frac{2^{u+1}}{2^k} (1+k-u) = k - \frac{k-u+1}{2^{k-u-1}} = k - \frac{2+(k-u-1)}{2^{k-u-1}} \geq k-1,$$

since  $(2+i)/2^i \leq 1$  for  $i \geq 2$ . ■