CS 573: Algorithms, Fall 2009

Homework 4, due Thursday, November 5, 23:59:59, 2009

Version 1.0

Name:	
Net ID:	Alias:

#	Score	Grader
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Neatly print your name(s), NetID(s), and the alias(es) you used for Homework 0 in the boxes above. Staple this sheet to the top of your homework. If you are on campus, submit the homework by submitting it in SC 3306 (or sliding it under the door).

> Shortly after the celebration of the four thousandth anniversary of the opening of space, Angary J. Gustible discovered Gustible's planet. The discovery turned out to be a tragic mistake.
> Gustible's planet was inhabited by highly intelligent life forms. They had moderate telepathic powers. They immediately mind-read Angary J. Gustible's entire mind and life history, and embarrassed him very deeply by making up an opera concerning his recent divorce.
> – From Gustible's Planet, Cordwainer Smith

Required Problems

1. Easy points.
   [20 Points]

   Let $S(k)$ be the depth of a sorting network with k inputs, and let $M(k)$ be the depth of a merging network with $2k$ inputs. Suppose that we have a sequence of n numbers to be sorted and we know that every number is within k positions of its correct position in the sorted order, which means that we need to move each number at most $(k-1)$ positions to sort the inputs. For example, in the sequence $3, 2, 1, 4, 5, 8, 7, 6, 9$, every number is within 3 positions of its correct position. But in sequence $3, 2, 1, 4, 5, 9, 8, 7, 6$, the number 9 and 6 are outside 3 positions of its correct position.

   Show that we can sort the n numbers in depth $S(k)+2M(k)$. (You need to prove your answer is correct.)

2. Matrix Madness
[20 Points]

We can sort the entries of an $m \times m$ matrix by repeating the following procedure k times:

1. Sort each odd-numbered row into monotonically increasing order.
2. Sort each even-numbered row into monotonically decreasing order.
3. Sort each column into monotonically increasing order.

(a) **[8 Points]** Suppose the matrix contains only 0's and 1's. We repeat the above procedure again and again until no changes occur. In what order should we read the matrix to obtain the sorted output ($m \times m$ numbers in increasing order)? Prove that any $m \times m$ matrix of 0's and 1's will be finally sorted.

(b) **[8 Points]** Prove that by repeating the above procedure, any matrix of real numbers can be sorted. [Hint:Refer to the proof of the zero-one principle.]

(c) **[4 Points]** Suppose k iterations are required for this procedure to sort the $m \times m$ numbers. Give an upper bound for k. The tighter your upper bound the better (prove you bound).

3. 3SUM
[20 Points]

Consider two sets A and B, each having n integers in the range from 0 to $10n$. We wish to compute the *Cartesian sum* of A and B, defined by

$$C = \{x + y : x \in A \text{ and } y \in B\}.$$

Note that the integers in C are in the range from 0 to $20n$. We want to find the elements of C and the number of times each element of C is realized as a sum of elements in A and B. Show that the problem can be solved in $O(n \lg n)$ time. (Hint: Use FFT.)

4. Common subsequence
[20 Points]

Given two sequences, $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ of real numbers, We want to determine whether there is an $i \geq 0$, such that $b_1 = a_{i+1}, b_2 = a_{i+2}, \ldots, b_m = a_{i+m}$. Show how to solve this problem in $O(n \log n)$ time with high probability. (Hint: Use FFT.)

5. Computing Polynomials Quickly.
[20 Points]

In the following, assume that given two polynomials $p(x), q(x)$ of degree at most n, one can compute the polynomial remainder of $p(x) \bmod q(x)$ in $O(n \log n)$ time. The ***remainder*** of $r(x) = p(x) \bmod q(x)$ is the unique polynomial of degree smaller than this of $q(x)$, such that $p(x) = q(x) * d(x) + r(x)$, where $d(x)$ is a polynomial.

Let $p(x) = \sum_{i=0}^{n-1} a_i x^i$ be a given polynomial.

(a) **[4 Points]** Prove that $p(x) \bmod (x - z) = p(z)$, for all z.

(b) **[4 Points]** We want to evaluate $p(\cdot)$ on the points $x_0, x_1, \ldots, x_{n-1}$. Let

$$P_{ij}(x) = \prod_{k=i}^{j}(x - x_k)$$

and

$$Q_{ij}(x) = p(x) \bmod P_{ij}(x).$$

Observe that the degree of Q_{ij} is at most $j - i$.

Prove that, for all x, $Q_{kk}(x) = p(x_k)$ and $Q_{0,n-1}(x) = p(x)$.

(c) **[4 Points]** Prove that for $i \leq k \leq j$, we have

$$\forall x \quad Q_{ik}(x) = Q_{ij}(x) \bmod P_{ik}(x)$$

and

$$\forall x \quad Q_{kj}(x) = Q_{ij}(x) \bmod P_{kj}(x).$$

(d) **[8 Points]** Given an $O(n \log^2 n)$ time algorithm to evaluate $p(x_0), \ldots, p(x_{n-1})$. Here $x_0, \ldots, x_{n-1}$ are n given real numbers.